

www.VNMATH.COM



APMO 1989 - 2009
PROBLEMS & SOLUTIONS

Editors
DONGPHD
SUUGAKU

© 2009
Powered by VnMath.Com

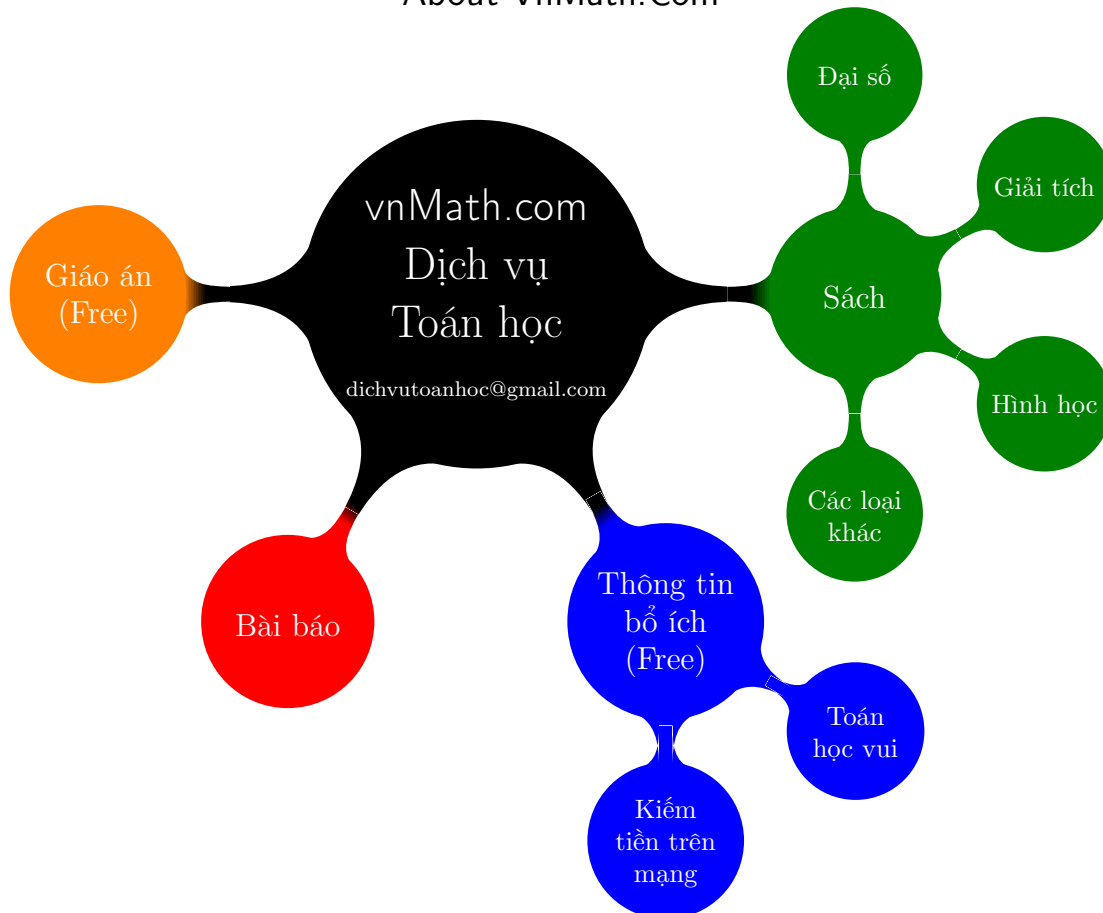
About Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO) started from 1989.

The APMO contest consists of one four-hour paper consisting of five questions of varying difficulty and each having a maximum score of 7 points. Contestants should not have formally enrolled at a university (or equivalent post-secondary institution) and they must be younger than 20 years of age on the 1st July of the year of the contest.

Every year, APMO is held in the afternoon of the second Monday of March for participating countries in the North and South Americas, and in the morning of the second Tuesday of March for participating countries on the Western Pacific and in Asia. The contest questions are to be collected from the contestants at the end of the APMO and are to be kept confidential until the Senior Coordinating Country posts them on the official APMO website. Additionally, each exam paper must contain a written legend, warning the students not to discuss the problems over the internet until that date.

About VnMath.Com



1st APMO 1989

A1. a_i are positive reals. $s = a_1 + \dots + a_n$. Prove that for any integer $n > 1$ we have $(1 + a_1) \dots (1 + a_n) < 1 + s + s^2/2! + \dots + s^n/n!$.

Solution

We use induction on n . For $n = 2$ the rhs is $1 + a_1 + a_2 + a_1 a_2 + (a_1^2 + a_2^2)/2 > \text{lhs}$. Assume the result is true for n . We note that, by the binomial theorem, for s and t positive we have $s^{m+1} + (m+1)t s^m < (s+t)^{m+1}$, and hence $s^{m+1}/(m+1)! + t s^m/m! < (s+t)^{m+1}/(m+1)!$. Summing from $m = 1$ to $n+1$ we get $(s+t) + (s^2/2! + t s/1!) + (s^3/3! + t s^2/2!) + \dots + (s^{n+1}/(n+1)! + t s^n/n!) < (s+t) + (s+t)^2/2! + \dots + (s+t)^{n+1}/(n+1)!$. Adding 1 to each side gives that $(1+t)(1 + s + s^2/2! + \dots + s^n/n!) < (1 + (s+t) + \dots + (s+t)^{n+1}/(n+1)!)$. Finally putting $t = a_{n+1}$ and using the result for n gives the result for $n+1$.

A2. Prove that $5n^2 = 36a^2 + 18b^2 + 6c^2$ has no integer solutions except $a = b = c = n = 0$.

Solution

The rhs is divisible by 3, so 3 must divide n . So $5n^2 - 36a^2 - 18b^2$ is divisible by 9, so 3 must divide c . We can now divide out the factor 9 to get: $5m^2 = 4a^2 + 2b^2 + 6d^2$. Now take m, a, b, d to be the solution with the smallest m , and consider residues mod 16. Squares = 0, 1, 4, or 9 mod 16. Clearly m is even so $5m^2 = 0$ or 4 mod 16. Similarly, $4a^2 = 0$ or 4 mod 16. Hence $2b^2 + 6d^2 = 0, 4$ or 12 mod 16. But $2b^2 = 0, 2$ or 8 mod 16 and $6d^2 = 0, 6$ or 8 mod 16. Hence $2b^2 + 6d^2 = 0, 2, 6, 8, 10$ or 14 mod 16. So it must be 0. So b and d are both even. So a cannot be even, otherwise $m/2, a/2, b/2, d/2$ would be a solution with smaller $m/2 < m$.

So we can divide out the factor 4 and get: $5k^2 = a^2 + 2e^2 + 6f^2$ with a odd. Hence k is also odd. So $5k^2 - a^2 = 4$ or 12 mod 16. But we have just seen that $2e^2 + 6f^2$ cannot be 4 or 12 mod 16. So there are no solutions.

A3. ABC is a triangle. X lies on the segment AB so that $AX/AB = 1/4$. CX intersects the median from A at A' and the median from B at B' . Points B'', C', A'', C'' are defined similarly. Find the area of the triangle $A''B''C''$ divided by the area of the triangle $A'B'C'$.

Solution

Answer: 25/49.

Let M be the midpoint of AB . We use vectors center G . Take $GA = \mathbf{A}, GB = \mathbf{B}, GC = \mathbf{C}$. Then $GM = \mathbf{A}/2 + \mathbf{B}/2$ and $GX = 3/4 \mathbf{A} + 1/4 \mathbf{C}$. Hence $GA' = 2/5 \mathbf{A}$ (showing it lies on GA) = $4/5 (3/4 \mathbf{A} + 1/4 \mathbf{B}) + 1/5 \mathbf{C}$, since $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$ (which shows it lies on CX). Similarly, $GB'' = 4/7 (1/2 \mathbf{A} + 1/2 \mathbf{C})$ (showing it lies on the median through B) = $2/7 \mathbf{A} + 2/7 \mathbf{C} = 5/7 (2/5 \mathbf{A}) + 2/7 \mathbf{C}$ (showing it lies on CA' and hence on CX). Hence $GB'' = -2/7 \mathbf{B}$. So we have shown that GB'' is parallel to GB' and $5/7$ the length. The same applies to the distances from the centroid to the other vertices. Hence triangle $A''B''C''$ is similar to triangle $A'B'C'$ and its area is 25/49 times the area of $A'B'C'$.

A4. Show that a graph with n vertices and k edges has at least $k(4k - n^2)/3n$ triangles.

Solution

Label the points $1, 2, \dots, n$ and let point i have degree d_i (no. of edges). Then if i and j are joined they have at least $d_i + d_j - 2$ other edges between them, and these edges join them to $n - 2$ other points. So there must be at least $d_i + d_j - n$ triangles which have i and j as two vertices. Hence the total number of triangles must be at least $\sum_{\text{edges } ij} (d_i + d_j - n)/3$. But $\sum_{\text{edges } ij} (d_i + d_j) = \sum d_i^2$, because each point i occurs in just d_i terms. Thus the total number of triangles is at least $(\sum d_i^2)/3 - nk/3$. But $\sum d_i^2 \geq (\sum d_i)^2/n$ (a special case of Chebyshev's inequality) $= 4k^2/n$. Hence result.

A5. f is a strictly increasing real-valued function on the reals. It has inverse f^{-1} . Find all possible f such that $f(x) + f^{-1}(x) = 2x$ for all x .

Solution

Answer: $f(x) = x + b$ for some fixed real b .

Suppose for some a we have $f(a) \neq a$. Then for some $b \neq 0$ we have $f(a) = a + b$. Hence $f(a + b) = a + 2b$ (because $f(f(a)) + f^{-1}(f(a)) = 2f(a)$, so $f(a + b) + a = 2a + 2b$) and by two easy inductions, $f(a + nb) = a + (n+1)b$ for all integers n (positive or negative).

Now take any x between a and $a + b$. Suppose $f(x) = x + c$. The same argument shows that $f(x + nc) = x + (n+1)c$. Since f is strictly increasing $x + c$ must lie between $f(a) = a + b$ and $f(a+b) = a + 2b$. So by a simple induction $x + nc$ must lie between $a + nb$ and $a + (n+1)b$. So c lies between $b + (x-a)/n$ and $b + (a+b-x)/n$ or all n . Hence $c = b$. Hence $f(x) = x + b$ for all x .

If there is no a for which $f(a) \neq a$, then we have $f(x) = x$ for all x .

2nd APMO 1990

A1. Given θ in the range $(0, \pi)$ how many (incongruent) triangles ABC have angle $A = \theta$, $BC = 1$, and the following four points concyclic: A , the centroid, the midpoint of AB and the midpoint of AC ?

Solution

Answer: 1 for $\theta \leq 60$ deg. Otherwise none.

Let O be the circumcenter of ABC and R the circumradius, let M be the midpoint of BC , and let G be the centroid. We may regard A as free to move on the circumcircle, whilst O , B and C remain fixed. Let X be the point on MO such that $MX/MO = 1/3$. An expansion by a factor 3, center M , takes G to A and X to O , so G must lie on the circle center X radius $R/3$.

The circle on diameter OA contains the midpoints of AB and AC (since if Z is one of the midpoints

OZ is perpendicular to the corresponding side). So if G also lies on this circle then angle OGA = 90 deg and hence angle MGO = 90 deg, so G must also lie on the circle diameter OM. Clearly the two circles for G either do not intersect in which case no triangle is possible which satisfies the condition or they intersect in one or two points. But if they intersect in two points, then corresponding triangles are obviously congruent (they just interchange B and C). So we have to find when the two circle intersect.

Let the circle center X meet the line OXM at P and Q with P on the same side of X as M. Now OM = R cos θ , so XM = $\frac{1}{3} R \cos \theta < \frac{1}{3} R = XP$, so M always lies inside PQ. Now XO = $\frac{2}{3} OM = \frac{1}{3} R (2 \cos \theta)$, so XQ = $\frac{1}{3} R > XO$ iff $2 \cos \theta < 1$ or $\theta > \pi/3$. Thus if $\theta > \pi/3$, then XQ > XO and so the circle diameter OM lies entirely inside the circle center X radius R/3 and so they cannot intersect. If $\theta = \pi/3$, then the circles touch at O, giving the equilateral triangle as a solution. If $\theta < \pi/3$, then the circles intersect giving one incongruent triangle satisfying the condition.

A2. x_1, \dots, x_n are positive reals. s_k is the sum of all products of k of the x_i (for example, if $n = 3$, $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_2x_3 + x_3x_1$, $s_3 = x_1x_2x_3$). Show that $s_k s_{n-k} \geq (nCk)^2 s_n$ for $0 < k < n$.

Solution

Each of s_k and s_{n-k} have nCk terms. So we may multiply out the product $s_k s_{n-k}$ to get a sum of $(nCk)^2$ terms. We now apply the arithmetic/geometric mean result. The product of all the terms must be a power of s_n by symmetry and hence must be s_n to the power of $(nCk)^2$. So the geometric mean of the terms is just s_n . Hence result.

A3. A triangle ABC has base AB = 1 and the altitude from C length h. What is the maximum possible product of the three altitudes? For which triangles is it achieved?

Solution

Answer: for $h \leq 1/2$, maximum product is h^2 , achieved by a triangle with right-angle at C; for $h > 1/2$, the maximum product is $h^3/(h^2 + 1/4)$, achieved by the isosceles triangle (AC = BC).

Solution by David Krumm

Let AC = b, BC = a, let the altitude from A have length x and the altitude from B have length y. Then $ax = by = h$, so $hxy = h^3/ab$. But $h = a \sin B$ and $b/\sin B = 1/\sin C$, so $h = ab \sin C$ and the product $hxy = h^2 \sin C$.

The locus of possible positions for C is the line parallel to AB and a distance h from it. [Or strictly the pair of such lines.] If $h \leq 1/2$, then there is a point on that line with angle ACB = 90 deg, so in this case we can obtain $hxy = h^2$ by taking angle ACB = 90 deg and that is clearly the best possible.

If $h > 1/2$, then there is no point on the line with angle ACB = 90 deg. Let L be the perpendicular bisector of AB and let L meet the locus at C. Then C is the point on the locus with the angle C a maximum. For if D is any other point of the line then the circumcircle of ABD also passes through the corresponding point D' on the other side of C and hence C lies inside the circumcircle. If L meets the circumcircle at C', then angle ADB = angle AC'B > angle ACB. Evidently $\sin C = 2 \sin C/2 \cos C/2 =$

$h/(h^2 + 1/4)$, so the maximum value of hxy is $h^3/(h^2 + 1/4)$.

My original, less elegant, solution is as follows.

Take AP perpendicular to AB and length h . Take Q to be on the line parallel to AB through P so that BQ is perpendicular to AB. Then C must lie on the line PQ (or on the corresponding line on the other side of AB). Let $a(A)$ be the length of the altitude from A to BC and $a(B)$ the length of the altitude from B to AC. If C maximises the product $h a(A) a(B)$, then it must lie on the segment PQ, for if angle ABC is obtuse, then both $a(A)$ and $a(B)$ are shorter than for ABQ. Similarly if BAC is obtuse. So suppose $PC = x$ with $0 \leq x \leq 1$. Then $AC = \sqrt{(x^2 + h^2)}$, so $a(B) = h/\sqrt{(x^2 + h^2)}$. Similarly, $a(A) = h/\sqrt{(1-x)^2 + h^2}$. So we wish to minimise $f(x) = (x^2 + h^2)((1-x)^2 + h^2) = x^4 - 2x^3 + (2h^2 + 1)x^2 - 2h^2x + h^4 + h^2$. We have $f'(x) = 2(2x-1)(x^2 - x + h^2)$, which has roots $x = 1/2, 1/2 \pm \sqrt{(1/4 - h^2)}$.

Thus for $h \geq 1/2$, the minimum is at $x = 1/2$, in which case $CA = CB$. For $h < 1/2$, the minimum is at $x = 1/2 \pm \sqrt{(1/4 - h^2)}$. But if M is the midpoint of AB and D is the point on AB with $AD = 1/2 \pm \sqrt{(1/4 - h^2)}$, then $DM = \sqrt{(1/4 - h^2)}$. But $DC = h$, and angle CDM = 90, so $MC = 1/2$ and hence angle ACB = 90.

A4. A graph with $n > 1$ points satisfies the following conditions: (1) no point has edges to all the other points, (2) there are no triangles, (3) given any two points A, B such that there is no edge AB, there is exactly one point C such that there are edges AC and BC. Prove that each point has the same number of edges. Find the smallest possible n .

Solution

Answer: 5.

We say A and B are joined if there is an edge AB. For any point X we write $\text{deg } X$ for the number of points joined to X. Take any point A. Suppose $\text{deg } A = m$. So there are m points B_1, B_2, \dots, B_m joined to A. No B_i, B_j can be joined for $i \neq j$, by (2), and a point $C \neq A$ cannot be joined to B_i and B_j for $i \neq j$, by (3). Hence there are $\text{deg } B_i - 1$ points C_{ij} joined to B_i and all the C_{ij} are distinct.

Now the only points that can be joined to C_{ij} , apart from B_i , are other C_{hk} , for by (3) any point of the graph is connected to A by a path of length 1 or 2. But C_{ij} cannot be joined to C_{ik} , by (2), and it cannot be joined to two distinct points C_{kh} and $C_{kh'}$ by (3), so it is joined to at most one point C_{kh} for each $k \neq i$. But by (3) there must be a point X joined to both B_k and C_{ij} (for $k \neq i$), and the only points joined to B_k are A and C_{kh} . Hence C_{ij} must be joined to at least one point C_{kh} for each $k \neq i$. Hence $\text{deg } C_{ij} = m$.

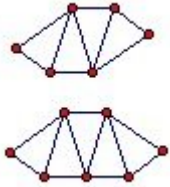
But now if we started with B_i instead of A and repeated the whole argument we would establish that $\text{deg } B_i$ is the same as the $\text{deg } C_{hk}$, where C_{hk} is one of the points joined to C_{i1} . Thus all the points have the same degree.

Suppose the degree of each point is m . Then with the notation above there is 1 point A, m points B_i and $m(m-1)$ points C_{jk} or $m^2 + 1$ in all. So $n = m^2 + 1$. The smallest possible m is 1, but that does not yield a valid graph because it does not satisfy (1). The next smallest possibility is $m = 2$, giving 5 points. It is easy to check that the pentagon satisfies all the conditions.

A5. Show that for any $n \geq 6$ we can find a convex hexagon which can be divided into n congruent triangles.

Solution

We use an isosceles triangle as the unit. The diagram shows $n = 4$ and $n = 5$. We can get any $n \geq 4$ by adding additional rhombi in the middle.



3rd APMO 1991

A1. ABC is a triangle. G is the centroid. The line parallel to BC through G meets AB at B' and AC at C' . Let A'' be the midpoint of BC , C'' the intersection of $B'C$ and BG , and B'' the intersection of $C'B$ and CG . Prove that $A''B''C''$ is similar to ABC .

Solution

Let M be the midpoint of AB and N the midpoint of AC . Let $A''M$ meet BG at X . Then X must be the midpoint of $A''M$ (an expansion by a factor 2 center B takes $A''M$ to CA and X to N). Also $BX/BN = 1/2$ and $BG/BN = 2/3$, so $XG = BX/3$. Let the ray CX meet AB at Z . Then $ZX = CX/3$. (There must be a neat geometric argument for this, but if we take vectors origin B , then $\mathbf{BX} = \mathbf{BN}/2 = \mathbf{BA}/4 + \mathbf{BC}/4$, so $\mathbf{BZ} = \mathbf{BA}/3$ and so $\mathbf{XZ} = 1/3 (\mathbf{BA}/4 - 3\mathbf{BC}/4) = \mathbf{CX}/3$.) So now triangles BXC and ZXG are similar, so ZG is parallel to BC , so Z is B' and X is C'' . But $A''X$ is parallel to AC and $1/4$ its length, so $A''C''$ is parallel to AC and $1/4$ its length. Similarly $A''B''$ is parallel to AB and $1/4$ its length. Hence $A''B''C''$ is similar to ABC .

A2. There are 997 points in the plane. Show that they have at least 1991 distinct midpoints. Is it possible to have exactly 1991 midpoints?

Solution

Answer: yes. Take the 997 points collinear at coordinates $x = 1, 3, \dots, 1993$. The midpoints are $2, 3, 4, \dots, 1992$.

Take two points A and B which are the maximum distance apart. Now consider the following midpoints: M , the midpoint of AB , the midpoint of each AX for any other X in the set (not A or B), and the midpoint of each BX . We claim that all these are distinct. Suppose X and Y are two other points (apart from A and B). Clearly the midpoints of AX and AY must be distinct (otherwise X and Y would coincide). Similarly the midpoints of BX and BY must be distinct. Equally, the midpoint of AX cannot be M (or X would coincide with B), nor can the midpoint of BX be M . Suppose, finally,

that N is the midpoint of AX and BY . Then $AYXB$ is a parallelogram and either AX or BY must exceed AB , contradicting the maximality of AB . So we have found 1991 distinct midpoints. The example above shows that there can be exactly 1991 midpoints.

A3. x_i and y_i are positive reals with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Show that $\sum_{i=1}^n x_i^2 / (x_i + y_i) \geq (\sum_{i=1}^n x_i) / 2$.

Solution

We use Cauchy-Schwartz: $\sum (x/\sqrt{x+y})^2 \sum (\sqrt{x+y})^2 \geq (\sum x)^2$. So $\sum x^2/(x+y) \geq (\sum x)^2 / (\sum(x+y)) = 1/2 \sum x$

A4. A sequence of values in the range $0, 1, 2, \dots, k-1$ is defined as follows: $a_1 = 1, a_n = a_{n-1} + n \pmod k$. For which k does the sequence assume all k possible values?

Solution

Let $f(n) = n(n+1)/2$, so $a_n = f(n) \pmod k$. If k is odd, then $f(n+k) = f(n) \pmod k$, so the sequence can only assume all possible values if a_1, \dots, a_k are all distinct. But $f(k-n) = f(n) \pmod k$, so there are at most $(k+1)/2$ distinct values. Thus k odd does not work.

If k is even, then $f(n+2k) = f(n) \pmod k$, so we need only look at the first $2k$ values. But $f((2k-1)-n) = f(n) \pmod k$ and $f(2k-1) = 0 \pmod k$, so the sequence assumes all values iff a_1, a_2, \dots, a_{k-1} assume all the values $1, 2, \dots, k-1$.

Checking the first few, we find $k = 2, 4, 8, 16$ work and $k = 6, 10, 12, 14$ do not. So this suggests that k must be a power of 2. Suppose k is a power of 2. If $f(r) = f(s) \pmod k$ for some $0 < r, s < k$, then $(r - s)(r + s + 1) = 0 \pmod k$. But each factor is $< k$, so neither can be divisible by k . Hence both must be even. But that is impossible (because their sum is $2r+1$ which is odd), so each of $f(1), f(2), \dots, f(k-1)$ must be distinct residues mod k . Obviously none can be $0 \pmod k$ (since $2k$ cannot divide $r(r+1)$ for $0 < r < k$ and so k cannot divide $f(r)$). Thus they must include all the residues $1, 2, \dots, k-1$. So k a power of 2 does work.

Now suppose h divides k and k works. If $f(n) = a \pmod k$, then $f(n) = a \pmod h$, so h must also work. Since odd numbers do not work, that implies that k cannot have any odd factors. So if k works it must be a power of 2.

A5. Circles C and C' both touch the line AB at B . Show how to construct all possible circles which touch C and C' and pass through A .

Solution

Take a common tangent touching C' at Q' and C at Q . Let the line from Q to A meet C again at P . Let the line from Q' to A meet C' again at P' . Let the C have center O and C' have center O' . Let the lines OP and $O'P'$ meet at X . Take X as the center of the required circle. There are two common tangents, so this gives two circles, one enclosing C and C' and one not.

To see that this construction works, invert wrt the circle on center A through B . C and C' go to

themselves under the inversion. The common tangent goes to a circle through A touching C and C'. Hence the point at which it touches C must be P and the point at which it touches C' must be P'.

4th APMO 1992

A1. A triangle has sides a, b, c . Construct another triangle sides $(-a + b + c)/2, (a - b + c)/2, (a + b - c)/2$. For which triangles can this process be repeated arbitrarily many times?

Solution

Answer: equilateral.

We may ignore the factor $1/2$, since clearly a triangle with sides x, y, z can be constructed iff a triangle with sides $2x, 2y, 2z$ can be constructed.

The advantage of considering the process as generating $(-a + b + c), (a - b + c), (a + b - c)$ from a, b, c is that the sum of the sides remains unchanged at $a + b + c$, so we can focus on just one of the three sides. Thus we are looking at the sequence $a, (a + b + c) - 2a, a + b + c - 2(-a + b + c), \dots$. Let $d = 2a - b - c$. We show that the process generates the sequence $a, a - d, a + d, a - 3d, a + 5d, a - 11d, a + 21d, \dots$. Let the n th term be $a + (-1)^n a_n d$. We claim that $a_{n+1} = 2a_n + (-1)^n$. This is an easy induction, for we have $a + (-1)^{n+1} a_{n+1} d = a + b + c - 2(a + (-1)^n a_n d)$ and hence $(-1)^{n+1} a_{n+1} d = -d - 2(-1)^n a_n d$, and hence $a_{n+1} = 2a_n + (-1)^n$. But this shows that a_n is unbounded. Hence if d is non-zero then the process ultimately generates a negative number. Thus a necessary condition for the process to generate triangles indefinitely is that $2a = b + c$. Similarly, $2b = c + a$ is a necessary condition. But these two equations imply (subtracting) $a = b$ and hence $a = c$. So a necessary condition is that the triangle is equilateral. But this is obviously also sufficient.

A2. Given a circle C centre O . A circle C' has centre X inside C and touches C at A . Another circle has centre Y inside C and touches C at B and touches C' at Z . Prove that the lines XB, YA and OZ are concurrent.

Solution

We need Ceva's theorem, which states that given points D, E, F on the lines BC, CA, AB , the lines AD, BE, CF are concurrent iff $(BD/DC)(CE/EA)(AF/FB) = 1$ (where we pay attention to the signs of BD etc, so that BD is negative if D lies on the opposite side of B to C). Here we look at the triangle OXY , and the points A on OX, B on OY and Z on XY (it is obvious that Z does lie on XY). We need to consider $(OA/AX)(XZ/ZY)(YB/BO)$. AX and BY are negative and the other distances positive, so the sign is plus. Also $OA = OB, AX = XZ$, and $ZY = YB$ (ignoring signs), so the expression is 1. Hence AY, XB and OZ are concurrent as required.

A3. Given three positive integers a, b, c , we can derive 8 numbers using one addition and one multiplication and using each number just once: $a+b+c, a+bc, b+ac, c+ab, (a+b)c, (b+c)a, (c+a)b, abc$. Show that if a, b, c are distinct positive integers such that $n/2 < a, b, c, \leq n$, then the 8 derived

numbers are all different. Show that if p is prime and $n \geq p^2$, then there are just $d(p-1)$ ways of choosing two distinct numbers b, c from $\{p+1, p+2, \dots, n\}$ so that the 8 numbers derived from p, b, c are *not* all distinct, where $d(p-1)$ is the number of positive divisors of $p-1$.

Solution

If $1 < a < b < c$, we have $a + b + c < ab + c < b + ac < a + bc$ and $(b+c)a < (a+c)b < (a+b)c < abc$. We also have $b + ac < (a+c)b$. So we just have to consider whether $a + bc = (b+c)a$. But if $a > c/2$, which is certainly the case if $n/2 < a, b, c \leq n$, then $a(b + c - 1) > c/2 (b + b) = bc$, so $a + bc < a(b + c)$ and all 8 numbers are different.

The numbers are not all distinct iff $p + bc = (b + c)p$. Put $b = p + d$. Then $c = p(p-1)/d + p$. Now we are assuming that $b < c$, so $p + d < p(p-1)/d + p$, hence $d^2 < p(p-1)$, so $d < p$. But p is prime so d cannot divide p , so it must divide $p-1$. So we get exactly $d(p-1)$ solutions provided that all the $c \leq n$. The largest c is that corresponding to $d = 1$ and is $p(p-1) + p = p^2 \leq n$.

A4. Find all possible pairs of positive integers (m, n) so that if you draw n lines which intersect in $n(n-1)/2$ distinct points and m parallel lines which meet the n lines in a further mn points (distinct from each other and from the first $n(n-1)/2$ points), then you form exactly 1992 regions.

Solution

Answer: $(1, 995), (10, 176), (21, 80)$.

n lines in general position divide the plane into $n(n+1)/2 + 1$ regions and each of the m parallel lines adds a further $n+1$ regions. So we require $n(n+1)/2 + 1 + m(n+1) = 1992$ or $(n+1)(2m+n) = 3982 = 2 \cdot 11 \cdot 181$. So $n+1$ must divide 3982, also $(n+1)n < 3982$, so $n \leq 62$. We are also told that n is positive. Thus $n = 0$ is disallowed. The remaining possibilities are $n+1 = 2, 11, 2 \cdot 11$. These give the three solutions shown above.

A5. $a_1, a_2, a_3, \dots, a_n$ is a sequence of non-zero integers such that the sum of any 7 consecutive terms is positive and the sum of any 11 consecutive terms is negative. What is the largest possible value for n ?

Solution

Answer: 16.

We cannot have 17 terms, because then:

$$\begin{aligned} a_1 + a_2 + \dots + a_{11} &< 0 \\ a_2 + a_3 + \dots + a_{12} &< 0 \\ a_3 + a_4 + \dots + a_{13} &< 0 \\ &\dots \\ a_7 + a_8 + \dots + a_{17} &< 0 \end{aligned}$$

So if we add the inequalities we get that an expression is negative. But notice that each column is positive. Contradiction.

On the other hand, a valid sequence of 16 terms is: -5, -5, 13, -5, -5, -5, 13, -5, -5, 13, -5, -5, -5, 13, -5,

-5. Any run of 7 terms has two 13s and five -5s, so sums to 1. Any run of 11 terms has three 13s and eight -5s, so sums to -1

5th APMO 1993

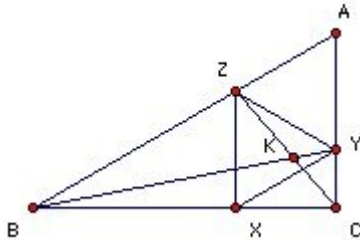
A1. A, B, C is a triangle. X, Y, Z lie on the sides BC, CA, AB respectively, so that AYZ and XYZ are equilateral. BY and CZ meet at K. Prove that $YZ^2 = YK \cdot YB$.

Solution

Use vectors. Take A as the origin. Let $\mathbf{AZ} = \mathbf{b}$, $\mathbf{AY} = \mathbf{c}$. We may take the equilateral triangles to have side 1, so $\mathbf{b}^2 = \mathbf{c}^2 = 1$ and $\mathbf{b} \cdot \mathbf{c} = 1/2$. Take \mathbf{AB} to be $k\mathbf{b}$. \mathbf{AX} is $\mathbf{b} + \mathbf{c}$, so \mathbf{AC} must be $k/(k-1)\mathbf{c}$ (then $\mathbf{AX} = 1/k(k\mathbf{b}) + (1 - 1/k)(k/(k-1)\mathbf{c})$, which shows that X lies on BC).

Hence $\mathbf{AK} = k/(k^2 - k + 1)(\mathbf{b} + (k-1)\mathbf{c})$. Writing this as $(k^2 - k)/(k^2 - k + 1)\mathbf{c} + 1/(k^2 - k + 1)(k\mathbf{b})$ shows that it lies on BY and writing it as $k/(k^2 - k + 1)\mathbf{b} + (k^2 - 2k + 1)/(k^2 - k + 1)(k/(k-1)\mathbf{c})$ shows that it lies on CZ. Hence $YK \cdot YB = \mathbf{YK} \cdot \mathbf{YB} = (k/(k^2 - k + 1)\mathbf{b} - 1/(k^2 - k + 1)\mathbf{c}) \cdot (k\mathbf{b} - \mathbf{c}) = (k\mathbf{b} - \mathbf{c})^2 / (k^2 - k + 1) = 1 = YZ^2$.

Thank to Achilles Porfyriadis for the following geometric proof



BZX and XYC are similar (sides parallel), so $BZ/ZX = XY/YC$. But XYZ is equilateral, so $BZ/ZY = ZY/YC$. Also $\angle BZY = \angle ZYC = 120^\circ$, so BZY and ZYC are similar. Hence $\angle ZBY = \angle YZC$. Hence YZ is tangent to the circle ZBK. Hence $YZ^2 = YK \cdot YB$

A2. How many different values are taken by the expression $[x] + [2x] + [5x/3] + [3x] + [4x]$ for real x in the range $0 \leq x \leq 100$?

Solution

Answer: 734.

Let $f(x) = [x] + [2x] + [3x] + [4x]$ and $g(x) = f(x) + [5x/3]$. Since $[y+n] = n + [y]$ for any integer n and real y, we have that $f(x+1) = f(x) + 10$. So for f it is sufficient to look at the half-open interval $[0, 1)$. f is obviously monotonic increasing and its value jumps at $x = 0, 1/4, 1/3, 1/2, 2/3, 3/4$. Thus f(x) takes 6 different values on $[0, 1)$.

$g(x+3) = g(x)$, so for g we need to look at the half-open interval $[0, 3)$. g jumps at the points at which f

jumps plus 4 additional points: $3/5, 1\ 1/5, 1\ 4/5, 2\ 2/5$. So on $[0, 3)$, $g(x)$ takes $3 \times 6 + 4 = 22$ different values. Hence on $[0, 99)$, $g(x)$ takes $33 \times 22 = 726$ different values. Then on $[99, 100]$ it takes a further $6 + 1 + 1$ (namely $g(99), g(99\ 1/4), g(99\ 1/3), g(99\ 1/2), g(99\ 3/5), g(99\ 2/3), g(99\ 3/4), g(100)$). Thus in total g takes $726 + 8 = 734$ different values.

A3. $p(x) = (x + a) q(x)$ is a real polynomial of degree n . The largest absolute value of the coefficients of $p(x)$ is h and the largest absolute value of the coefficients of $q(x)$ is k . Prove that $k \leq hn$.

Solution

Let $p(x) = p_0 + p_1x + \dots + p_nx^n$, $q(x) = q_0 + q_1x + \dots + q_{n-1}x^{n-1}$, so $h = \max |p_i|$, $k = \max |q_i|$.

If $a = 0$, then the result is trivial. So assume a is non-zero. We have $p_n = q_{n-1}$, $p_{n-1} = q_{n-2} + aq_{n-1}$, $p_{n-2} = q_{n-3} + aq_{n-2}, \dots, p_1 = q_0 + aq_1, p_0 = aq_0$.

We consider two cases. Suppose first that $|a| \geq 1$. Then we show by induction that $|q_i| \leq (i+1)h$. We have $q_0 = p_0/a$, so $|q_0| \leq h$, which establishes the result for $i = 0$. Suppose it is true for i . We have $q_{i+1} = (p_{i+1} - q_i)/a$, so $|q_{i+1}| \leq |p_{i+1}| + |q_i| \leq h + (i+1)h = (i+2)h$, so it is true for $i+1$. Hence it is true for all $i < n$. So $k \leq \max(h, 2h, \dots, nh) = nh$.

The remaining possibility is $0 < |a| < 1$. In this case we show by induction that $|q_{n-i}| \leq ih$. We have $q_{n-1} = p_n$, so $|q_{n-1}| \leq |p_n| \leq h$, which establishes the result for $i = 1$. Suppose it is true for i . We have $q_{n-i-1} = p_{n-i} - aq_{n-i}$, so $|q_{n-i-1}| + |q_{n-i}| \leq h + ih = (i+1)h$, so it is true for $i+1$. Hence it is true for all $1 \leq i \leq n$. Hence $k \leq \max(h, 2h, \dots, nh) = nh$.

A4. Find all positive integers n for which $x^n + (x+2)^n + (2-x)^n = 0$ has an integral solution.

Solution

Answer: $n = 1$.

There are obviously no solutions for even n , because then all terms are non-negative and at least one is positive. $x = -4$ is a solution for $n = 1$. So suppose n is odd n and > 3 .

If x is positive, then $x^n + (x+2)^n > (x+2)^n > (x-2)^n$, so $x^n + (x+2)^n + (2-x)^n > 0$. Hence any solution x must be negative. Put $x = -y$. Clearly $x = -1$ is not a solution for any n , so if $x = -y$ is a solution then $(x+2) = -(y-2) \leq 0$ we have $(y+2)^n = y^n + (y-2)^n$. Now $4 = ((y+2) - (y-2))$ divides $(y+2)^n - (y-2)^n$. Hence 2 divides y . Put $y = 2z$, then we have $(z+1)^n = z^n + (z-1)^n$. Now 2 divides $(z+1)^n - (z-1)^n$ so 2 divides z , so $z+1$ and $z-1$ are both odd. But $a^n - b^n = (a - b)(a^{n-1}n-2b + a^{n-3}b^2 + \dots + b^{n-1})$. If a and b are both odd, then each term in $(a^{n-1}n-2b + a^{n-3}b^2 + \dots + b^{n-1})$ is odd and since n is odd there are an odd number of terms, so $(a^{n-1}n-2b + a^{n-3}b^2 + \dots + b^{n-1})$ is odd. Hence, putting $a=z+1, b=z-1$, we see that $(z+1)^n - (z-1)^n = 2(a^{n-1}n-2b + a^{n-3}b^2 + \dots + b^{n-1})$ is not divisible by 4 . But it equals z^n with z even. Hence n must be 1 .

A5. C is a 1993 -gon of lattice points in the plane (not necessarily convex). Each side of C has no lattice points except the two vertices. Prove that at least one side contains a point (x, y) with $2x$ and $2y$ both odd integers.

Solution

We consider the midpoint of each side. We say that a vertex (x, y) is pure if x and y have the same parity and impure if x and y have opposite parity. Since the total number of vertices is odd, there must be two adjacent pure vertices P and Q or two adjacent impure vertices P and Q . But in either case the midpoint of P and Q either has both coordinates integers, which we are told does not happen, or as both coordinates of the form an integer plus half, which therefore must occur.

6th APMO 1994

A1. Find all real-valued functions f on the reals such that (1) $f(1) = 1$, (2) $f(-1) = -1$, (3) $f(x) \leq f(0)$ for $0 < x < 1$, (4) $f(x + y) \geq f(x) + f(y)$ for all x, y , (5) $f(x + y) \leq f(x) + f(y) + 1$ for all x, y .

Solution

Answer: $f(x) = [x]$.

$f(x+1) \geq f(x) + f(1) = f(x) + 1$ by (4) and (1). But $f(x) \geq f(x+1) + f(-1) = f(x+1) - 1$ by (4) and (2). Hence $f(x+1) = f(x) + 1$.

In particular, $1 = f(1) = f(0+1) = f(0) + 1$, so $f(0) = 0$. Hence, by (3), $f(x) \leq 0$ for $0 < x < 1$. But, by (5), $1 = f(1) = f(x + 1-x) \leq f(x) + f(1-x) + 1$, so $f(x) + f(1-x) \geq 0$. But if $0 < x < 1$, then also $0 < 1-x < 1$, so $f(x) = f(1-x) = 0$.

Thus we have established that $f(x) = 0$ for $0 \leq x < 1$, and $f(x+1) = f(x) + 1$. It follows that $f(x) = [x]$ for all x .

A2. ABC is a triangle and A, B, C are not collinear. Prove that the distance between the orthocenter and the circumcenter is less than three times the circumradius.

Solution

We use vectors. It is well-known that the circumcenter O , the centroid G and the orthocenter H lie on the Euler line and that $\overrightarrow{OH} = 3 \overrightarrow{OG}$. Hence taking vectors with origin O , $\overrightarrow{OH} = 3 \overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Hence $|\overrightarrow{OH}| \leq |\overrightarrow{OA}| + |\overrightarrow{OB}| + |\overrightarrow{OC}| = 3 \times \text{circumradius}$. We could have equality only if ABC were collinear, but that is impossible, because ABC would not then be a triangle.

A3. Find all positive integers n such that $n = a^2 + b^2$, where a and b are relatively prime positive integers, and every prime not exceeding \sqrt{n} divides ab .

Solution

Answer: $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, $13 = 2^2 + 3^2$.

The key is to use the fact that a and b are relatively prime. We show in fact that they must differ by 1 (or 0). Suppose first that $a = b$. Then since they are relatively prime they must both be 1. That gives the first answer above. So we may take $a > b$. Then $(a - b)^2 < a^2 + b^2 = n$, so if $a - b$ is not 1, it must have a prime factor which divides ab . But then it must divide a or b and hence both. Contradiction. So $a = b + 1$.

Now $(b - 1)^2 < b^2 < n$, so any prime factor p of $b - 1$ must divide $ab = b(b + 1)$. It cannot divide b (or it would divide b and $b - 1$ and hence 1), so it must divide $b + 1$ and hence must be 2. But if 4 divides $b - 1$, then 4 does not divide $b(b - 1)$, so $b - 1$ must be 0, 1 or 2. But it is now easy to check the cases $a, b = (4, 3), (3, 2), (2, 1)$.

A4. Can you find infinitely many points in the plane such that the distance between any two is rational and no three are collinear?

Solution

Answer: yes.

Let $\theta = \cos^{-1}3/5$. Take a circle center O radius 1 and a point X on the circle. Take P_n on the circle such that angle $XOP_n = 2n\theta$. We establish (A) that the P_n are all distinct and (B) that the distances P_mP_n are all rational.

We establish first that we can express $2 \cos nx$ as a polynomial of degree n in $(2 \cos x)$ with integer coefficients and leading coefficient 1. For example, $2 \cos 2x = (2 \cos x)^2 - 1$, $2 \cos 3x = (2 \cos x)^3 - 3(2 \cos x)$. We proceed by induction. Let the polynomial be $p_n(2 \cos x)$. We have that $p_1(y) = y$ and $p_2(y) = y^2 - 1$. Suppose we have found p_m for $m \leq n$. Now $\cos(n+1)x = \cos nx \cos x - \sin nx \sin x$, and $\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$, so $\cos(n+1)x = 2 \cos x \cos nx - \cos(n-1)x$. Hence $p_{n+1}(y) = y p_n(y) - p_{n-1}(y)$. Hence the result is also true for $n+1$.

It follows that (1) if $\cos x$ is rational, then so is $\cos nx$, and (2) if $\cos x$ is rational, then x/π is irrational. To see (2), suppose that $x/\pi = m/n$, with m and n integers. Then nx is a multiple of π and hence $\cos nx = 1$, so $p_n(2 \cos x) = 0$. Now we may write $p_n(y) = y^n + a_{n-1}y^{n-1} + \dots + a_0$. Now if also $\cos x = r/s$, with r and s relatively prime integers, then we have, $p_n(2 \cos x) = r^n + a_{n-1}s r^{n-1} + \dots + a_0s^n = 0$. But now s divides all terms except the first. Contradiction.

Thus we cannot have $\cos m\theta = \cos n\theta$ for any distinct integers m, n , for then θ/π would be rational as well as $\cos \theta$. So we have established (A).

We have also established that all $\cos n\theta$ are rational. But since $\sin(n+1)x = \sin nx \cos x + \cos nx \sin x$ and $\sin \theta = 4/5$, it is a trivial induction that all $\sin n\theta$ are also rational. Now $P_mP_n = 2 |\sin(m - n)\theta|$, so all the distances P_mP_n are rational, thus establishing (B).

A5. Prove that for any $n > 1$ there is either a power of 10 with n digits in base 2 or a power of 10 with n digits in base 5, but not both.

Solution

10^k has n digits in base 5 iff $5^{n-1} < 10^k < 5^n$. Similarly, 10^h has n digits in base 2 iff $2^{n-1} < 10^h < 2^n$. So if we can find both 10^k with n digits in base 5 and 10^h with n digits in base 2, then, multiplying the two inequalities, we have $10^{n-1} < 10^{h+k} < 10^n$, which is clearly impossible. This establishes the "but not both" part.

Let S be the set of all positive powers of 2 or 5. Order the members of S in the usual way and let a_n be the n -th member of the set. We claim that if $a_n = 2^k$, then 10^k has n digits in base 5, and if $a_n = 5^h$, then 10^h has n digits in base 2. We use induction on n .

$a_2 = 2^1$, $a_3 = 2^2$, $a_4 = 5^1$, $a_5 = 2^3$, Thus the claim is certainly true for $n = 2$. Suppose it is true for n .

Note that 10^k has n digits in base 5 iff $5^{n-k-1} < 2^k < 5^{n-k}$. Similarly, 10^h has n digits in base 2 iff $2^{n-h-1} < 5^h < 2^{n-h}$. There are 3 cases. Case (1). $a_n = 2^k$ and $a_{n+1} = 2^{k+1}$. Hence 10^{k+1} has $n+1$ digits in base 5. Case (2). $a_n = 2^k$ and a_{n+1} is a power of 5. Hence a_{n+1} must be 5^{n-k} . Hence $2^k < 5^{n-k} < 2^{k+1}$. Hence $2^n < 10^{n-k} < 2^{n+1}$. So 10^{n-k} has $n+1$ digits in base 2. Case (3). $a_n = 5^h$. Since there is always a power of 2 between two powers of 5, a_{n+1} must be a power of 2. Hence it must be 2^{n-h} . So we have $5^h < 2^{n-h} < 5^{h+1}$. So $5^n < 10^{n-h} < 5^{n+1}$ and hence 10^{n-h} has $n+1$ digits in base 5.

Jacob Tsimmerman pointed out that the second part can be done in a similar way to the first - which is neater than the above:

If no power of 10 has n digits in base 2 or 5, then for some h, k : $10^h < 2^{n-1} < 2^n < 10^{h+1}$ and $10^k < 5^{n-1} < 5^n < 10^{k+1}$. Hence $10^{h+k} < 10^{n-1} < 10^n < 10^{h+k+2}$. But there is only one power of 10 between $h+k$ and $h+k+2$.

7th APMO 1995

A1. Find all real sequences $x_1, x_2, \dots, x_{1995}$ which satisfy $2\sqrt{(x_n - n + 1)} \geq x_{n+1} - n + 1$ for $n = 1, 2, \dots, 1994$, and $2\sqrt{(x_{1995} - 1994)} \geq x_1 + 1$.

Solution

Answer: the only such sequence is $1, 2, 3, \dots, 1995$.

Put $x_{1995} = 1995 + k$. We show by induction (moving downwards from 1995) that $x_n \geq n + k$. For suppose $x_{n+1} \geq n + k + 1$, then $4(x_n - n + 1) \geq (x_{n+1} - n + 1)^2 \geq (k+2)^2 \geq 4k + 4$, so $x_n \geq n + k$. So the result is true for all $n \geq 1$. In particular, $x_1 \geq 1 + k$. Hence $4(x_{1995} - 1994) = 4(1 + k) \geq (2 + k)^2 = 4 + 4k + k^2$, so $k^2 \leq 0$, so $k = 0$.

Hence also $x_n \geq n$ for $n = 1, 2, \dots, 1994$. But now if $x_n = n + k$, with $k > 0$, for some $n < 1995$, then the same argument shows that $x_1 \geq 1 + k$ and hence $4 = 4(x_{1995} - 1994) \geq (x_1 + 1)^2 \geq (2 + k)^2 = 4 + 4k + k^2 > 4$. Contradiction. Hence $x_n = n$ for all $n \leq 1995$.

A2. Find the smallest n such that any sequence a_1, a_2, \dots, a_n whose values are relatively prime square-free integers between 2 and 1995 must contain a prime. [An integer is square-free if it is not divisible by any square except 1.]

Solution

Answer: $n = 14$.

We can exhibit a sequence with 13 terms which does not contain a prime: $2 \cdot 101 = 202$, $3 \cdot 97 = 291$, $5 \cdot 89 = 445$, $7 \cdot 83 = 581$, $11 \cdot 79 = 869$, $13 \cdot 73 = 949$, $17 \cdot 71 = 1207$, $19 \cdot 67 = 1273$, $23 \cdot 61 = 1403$, $29 \cdot 59 = 1711$, $31 \cdot 53 = 1643$, $37 \cdot 47 = 1739$, $41 \cdot 43 = 1763$. So certainly $n \geq 14$.

If there is a sequence with $n \geq 14$ not containing any primes, then since there are only 13 primes not exceeding 41, at least one member of the sequence must have at least two prime factors exceeding 41. Hence it must be at least $43 \cdot 47 = 2021$ which exceeds 1995. So $n = 14$ is impossible.

A3. ABCD is a fixed cyclic quadrilateral with AB not parallel to CD. Find the locus of points P for which we can find circles through AB and CD touching at P.

Solution

Answer: Let the lines AB and CD meet at X. Let R be the length of a tangent from X to the circle ABCD. The locus is the circle center X radius R. [Strictly you must exclude four points unless you allow the degenerate straight line circles.]

Let X be the intersection of the lines AB and CD. Let R be the length of a tangent from X to the circle ABCD. Let C_0 be the circle center X radius R. Take any point P on C_0 . Then considering the original circle ABCD, we have that $R^2 = XA \cdot XB = XC \cdot XD$, and hence $XP^2 = XA \cdot XB = XC \cdot XD$.

If C_1 is the circle through C, D and P, then $XC \cdot XD = XP^2$, so XP is tangent to the circle C_1 . Similarly, the circle C_2 through A, B and P is tangent to XP. Hence C_1 and C_2 are tangent to each other at P. Note that if P is one of the 4 points on AB or CD and C_0 , then this construction does not work unless we allow the degenerate straight line circles AB and CD.

So we have established that all (or all but 4) points of C_0 lie on the locus. But for any given circle through C, D, there are only two circles through A, B which touch it (this is clear if you consider how the circle through A, B changes as its center moves along the perpendicular bisector of AB), so there are at most 2 points on the locus lying on a given circle through C, D. But these are just the two points of intersection of the circle with C_0 . So there are no points on the locus not on C_0 .

A4. Take a fixed point P inside a fixed circle. Take a pair of perpendicular chords AC, BD through P. Take Q to be one of the four points such that AQBP, BQCP, CQDP or DQAP is a rectangle. Find the locus of all possible Q for all possible such chords.

Solution

Let O be the center of the fixed circle and let X be the center of the rectangle ASCQ. By the cosine

rule we have $OQ^2 = OX^2 + XQ^2 - 2 \cdot OX \cdot XQ \cos \theta$ and $OP^2 = OX^2 + XP^2 - 2 \cdot OX \cdot XP \cos(\theta + \pi)$, where θ is the angle OXQ . But $\cos(\theta + \pi) = -\cos \theta$, so $OQ^2 + OP^2 = 2OX^2 + 2XQ^2$. But since X is the center of the rectangle $XQ = XC$ and since X is the midpoint of AC , OX is perpendicular to AC and hence $XO^2 + XC^2 = OC^2$. So $OQ^2 = 2OC^2 - OP^2$. But this quantity is fixed, so Q must lie on the circle center O radius $\sqrt{(2R^2 - OP^2)}$, where R is the radius of the circle.

Conversely, it is easy to see that all points on this circle can be reached. For given a point Q on the circle radius $\sqrt{(2R^2 - OP^2)}$ let X be the midpoint of PQ . Then take the chord AC to have X as its midpoint.

A5. f is a function from the integers to $\{1, 2, 3, \dots, n\}$ such that $f(A)$ and $f(B)$ are unequal whenever A and B differ by 5, 7 or 12. What is the smallest possible n ?

Solution

Answer: $n = 4$.

Each pair of 0, 5, 12 differ by 5, 7 or 12, so $f(0), f(5), f(12)$ must all be different, so $n \geq 3$.

We can exhibit an f with $n = 4$. Define $f(m) = 1$ for $m = 1, 3, 5, 7, 9, 11 \pmod{24}$, $f(m) = 2$ for $m = 2, 4, 6, 8, 10, 12 \pmod{24}$, $f(m) = 3$ for $m = 13, 15, 17, 19, 21, 23 \pmod{24}$, $f(m) = 4$ for $m = 14, 16, 18, 20, 22, 0 \pmod{24}$.

8th APMO 1996

A1. $ABCD$ is a fixed rhombus. P lies on AB and Q on BC , so that PQ is perpendicular to BD . Similarly P' lies on AD and Q' on CD , so that $P'Q'$ is perpendicular to BD . The distance between PQ and $P'Q'$ is more than $BD/2$. Show that the perimeter of the hexagon $APQCQ'P'$ depends only on the distance between PQ and $P'Q'$.

Solution

BPQ and $DQ'P'$ are similar. Let PQ meet BD at X and $P'Q'$ meet BD at Y . XY is fixed, so $BX + DY$ is fixed. Hence also, $BP + DQ'$ and $BQ + DP'$ and $PQ + P'Q'$ are fixed. So $PQ + P'Q' - BP - BQ - DP' - DQ'$ is fixed, so $PQ + P'Q' + (AB - BP) + (BC - BQ) + (CD - DP') + (DA - DQ')$ is fixed, and that is the perimeter of the hexagon.

A2. Prove that $(n+1)^m n^m \geq (n+m)! / (n-m)! \geq 2^m m!$ for all positive integers n, m with $n \geq m$.

Solution

For any integer $k \geq 1$, we have $(n+k)(n-k+1) = n^2 + n - k^2 + k \leq n(n+1)$. Taking the product from $k = 1$ to m we get $(n+m)! / (n-m)! \leq (n+1)^m n^m$.

For $k = 1, 2, \dots, m$, we have $n \geq k$ and hence $n+k \geq 2k$. Taking the product from $k = 1$ to m , we get

$$(n + m)!/(n - m)! \geq 2^m m! .$$

A3. Given four concyclic points. For each subset of three points take the incenter. Show that the four incenters form a rectangle.

Solution

Take the points as A, B, C, D in that order. Let I be the incenter of ABC. The ray CI bisects the angle ACB, so it passes through M, the midpoint of the arc AB. Now $\angle MBI = \angle MBA + \angle IBA = \angle MCA + \angle IBA = (\angle ACB + \angle ABC)/2 = 90^\circ - (\angle CAB) /2 = 90^\circ - \angle CMB/2 = 90^\circ - \angle IMB/2$. So the bisector of $\angle IMB$ is perpendicular to IB. Hence MB = MI. Let J be the incenter of ABD. Then similarly MA = MJ. But MA = MB, so the four points A, B, I, J are concyclic (they lie on the circle center M). Hence $\angle BIJ = 180^\circ - \angle BAJ = 180^\circ - \angle BAD/2$.

Similarly, if K is the incenter of ADC, then $\angle BJK = 180^\circ - \angle BDC/2$. Hence $\angle IJK = 360^\circ - \angle BIJ - \angle BJK = (180^\circ - \angle BIJ) + (180^\circ - \angle BJK) = (\angle BAD + \angle BDC)/2 = 90^\circ$. Similarly, the other angles of the incenter quadrilateral are 90° , so it is a rectangle.

A4. For which n in the range 1 to 1996 is it possible to divide n married couples into exactly 17 single sex groups, so that the size of any two groups differs by at most one.

Solution

Answer: 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 36, 37, 38, 39, 40, 45, 46, 47, 48, 54, 55, 56, 63, 64, 72.

If $n = 17k$, then the group size must be $2k$. Hence no arrangement is possible, because one sex has at most 8 groups and $8 \cdot 2k < n$.

If $2n = 17k+h$ with $0 < h < 17$, then the group size must be k or $k+1$. One sex has at most 8 groups, so $8(k+1) \geq n$. Hence $16k + 16 \geq 17k + h$, so $16 - h \geq k$ (*). We also require that $9k \leq n$. Hence $18k < 2n = 17k + h$, so $k \leq h$ (**). With (*) this implies that $k \leq 8$. So $n \leq 75$.

Each group has at least one person, so we certainly require $n \geq 9$ and hence $k \geq 1$. It is now easiest to enumerate. For $k = 1$, we can have $h = 1, 3, \dots, 15$, giving $n = 9-16$. For $k = 2$, we can have $h = 2, 4, \dots, 14$, giving $n = 18-24$. For $k = 3$, we can have $h = 3, 5, \dots, 13$, giving $n = 27-32$. For $k = 4$, we can have $h = 4, 6, \dots, 12$, giving $n = 36-40$. For $k = 5$ we can have $h = 5, 7, 9, 11$, giving $n = 45-48$. For $k = 6$, we can have $h = 6, 8, 10$, giving $n = 54, 55, 56$. For $k = 7$, we can have $h = 7, 9$, giving $n = 63, 64$. For $k = 8$, we can have $h = 8$, giving $n = 72$.

A5. A triangle has side lengths a, b, c. Prove that $\sqrt{(a + b - c)} + \sqrt{(b + c - a)} + \sqrt{(c + a - b)} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$. When do you have equality?

Solution

Let $A^2 = b + c - a$, $B^2 = c + a - b$, $C^2 = a + b - c$. Then $A^2 + B^2 = 2c$. Also $A = B$ iff $a = b$. We have $(A$

- $B)^2 \geq 0$, with equality iff $A = B$. Hence $A^2 + B^2 \geq 2AB$ and so $2(A^2 + B^2) \geq (A + B)^2$ or $4c \geq (A + B)^2$ or $2\sqrt{c} \geq A + B$, with equality iff $A = B$. Adding the two similar relations we get the desired inequality, with equality iff the triangle is equilateral

9th APMO 1997

A1. Let $T_n = 1 + 2 + \dots + n = n(n+1)/2$. Let $S_n = 1/T_1 + 1/T_2 + \dots + 1/T_n$. Prove that $1/S_1 + 1/S_2 + \dots + 1/S_{1996} > 1001$.

Solution

$1/T_m = 2(1/m - 1/(m+1))$. Hence $S_n/2 = 1 - 1/(n+1)$. So $1/S_n = (1 + 1/n)/2$. Hence $1/S_1 + 1/S_2 + \dots + 1/S_n = 1996/2 + (1 + 1/2 + 1/3 + \dots + 1/1996)/2$.

Now $1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + \dots + 1/16) + (1/17 + \dots + 1/32) + (1/33 + \dots + 1/64) + (1/65 + \dots + 1/128) + (1/129 + \dots + 1/256) + (1/257 + \dots + 1/512) + (1/513 + \dots + 1/1024) > 1 + 1/2 + 1/2 + \dots + 1/2 = 6$. So $1/S_1 + 1/S_2 + \dots + 1/S_n = 1996/2 + 6/2 = 998 + 3 = 1001$.

A2. Find an n in the range $100, 101, \dots, 1997$ such that n divides $2^n + 2$.

Solution

Answer: the only such number is 946.

We have $2^{p-1} = 1 \pmod p$ for any prime p , so if we can find h in $\{1, 2, \dots, p-2\}$ for which $2^h = -2 \pmod p$, then $2^k = -2 \pmod p$ for any $h = k \pmod p$. Thus we find that $2^k = -2 \pmod 5$ for $k = 3 \pmod 4$, and $2^k = -2 \pmod 11$ for $k = 6 \pmod 10$. So we might then hope that $5 \cdot 11 = 3 \pmod 4$ and $= 6 \pmod 10$. Unfortunately, it does not! But we try searching for more examples.

The simplest would be to look at pq . Suppose first that p and q are both odd, so that pq is odd. If $k = h \pmod{p-1}$, then we need h to be odd (otherwise pq would have to be even). So the first step is to get a list of primes p with $2^h = -2 \pmod p$ for some odd $h < p$. We always have $2^{p-1} = 1 \pmod p$, so we *sometimes* have $2^{(p-1)/2} = -1 \pmod p$ and hence $2^{(p+1)/2} = -2 \pmod p$. If $(p+1)/2$ is to be odd then $p = 1 \pmod 4$. So searching such primes we find $3 \pmod 5, 7 \pmod 13, 15 \pmod 29, 19 \pmod 37, 27 \pmod 53, 31 \pmod 61$. We require pq to lie in the range $100-1997$, so we check $5 \cdot 29$ (not $= 3 \pmod 4$), $5 \cdot 37$ (not $= 3 \pmod 4$), $5 \cdot 53$ (not $= 3 \pmod 4$), $5 \cdot 61$ (not $= 3 \pmod 4$), $13 \cdot 29$ (not $= 7 \pmod 12$), $13 \cdot 37$ (not $= 7 \pmod 12$), $13 \cdot 53$ (not $= 7 \pmod 12$), $13 \cdot 61$ (not $= 7 \pmod 12$), $29 \cdot 37$ (not $= 15 \pmod 28$), $29 \cdot 53$ (not $= 15 \pmod 28$), $29 \cdot 61$ (not $= 15 \pmod 28$), $37 \cdot 53$ (not $= 19 \pmod 36$). So that does not advance matters much!

$2p$ will not work (at least with $h = (p+1)/2$) because we cannot have $2p = (p+1)/2 \pmod{p-1}$. So we try looking at $2pq$. This requires that p and $q = 3 \pmod 4$. So searching for suitable p we find $6 \pmod 11, 10 \pmod 19, 22 \pmod 43, 30 \pmod 59, 34 \pmod 67, 42 \pmod 83$. So we look at $2 \cdot 11 \cdot 43 = 946$, which works.

Proving that it is unique is harder. The easiest way is to use a computer to search (approx 5 min to write a Maple program or similar and a few seconds to run it).

A3. ABC is a triangle. The bisector of A meets the segment BC at X and the circumcircle at Y. Let $r_A = AX/AY$. Define r_B and r_C similarly. Prove that $r_A/\sin^2 A + r_B/\sin^2 B + r_C/\sin^2 C \geq 3$ with equality iff the triangle is equilateral.

Solution

$AX/AB = \sin B/\sin AXB = \sin B/\sin(180 - B - A/2) = \sin B/\sin(B + A/2)$. Similarly, $AB/AY = \sin AYB/\sin ABY = \sin C/\sin(B + CBY) = \sin C/\sin(B + A/2)$. So $AX/AY = \sin B \sin C/\sin^2(B + A/2)$. Hence $r_A/\sin^2 A = s_A/\sin^2(B + A/2)$, where $s_A = \sin B \sin C/\sin^2 A$. Similarly for r_B and r_C . Now $s_A s_B s_C = 1$, so the arithmetic/geometric mean result gives $s_A + s_B + s_C \geq 3$. But $1/\sin k \geq 1$ for any k , so $r_A/\sin^2 A + r_B/\sin^2 B + r_C/\sin^2 C \geq 3$.

A necessary condition for equality is that $\sin^2(B + A/2) = \sin^2(B + A/2) = \sin^2(B + A/2) = 1$ and hence $A = B = C$. But it is easily checked that this is also sufficient.

A4. P_1 and P_3 are fixed points. P_2 lies on the line perpendicular to P_1P_3 through P_3 . The sequence P_4, P_5, P_6, \dots is defined inductively as follows: P_{n+1} is the foot of the perpendicular from P_n to $P_{n-1}P_{n-2}$. Show that the sequence converges to a point P (whose position depends on P_2). What is the locus of P as P_2 varies?

Solution

$P_n P_{n+1} P_{n+2}$ lies inside $P_{n-1} P_n P_{n+1}$. So we have sequence of nested triangles whose size shrinks to zero. Each triangle is a closed set, so there is just one point P in the intersection of all the triangles and it is clear that the sequence P_n converges to it.

Obviously all the triangles $P_n P_{n+1} P_{n+2}$ are similar (but not necessarily with the vertices in that order). So P must lie in the same position relative to each triangle and we must be able to obtain one triangle from another by rotation and expansion about P . In particular, $P_5 P_4 P_6$ is similar (with the vertices in that order) to $P_1 P_2 P_3$, and $P_4 P_5$ is parallel to $P_1 P_2$, so the rotation to get one from the other must be through π and P must lie on $P_1 P_5$. Similarly $P_3 P_4 P_5$ must be obtained from $P_1 P_2 P_3$ by rotation about P through $\pi/2$ and expansion. But this takes $P_1 P_5$ into a perpendicular line through P_3 . Hence $P_1 P$ is perpendicular to $P_3 P$. Hence P lies on the circle diameter $P_1 P_3$.

However, not all points on this circle are points of the locus. $P_3 P_5 = P_3 P_4 \cos P_1 = P_3 P_1 \sin P_1 \cos P_2 = 1/2 P_3 P_1 \sin 2P_1$, so we can obtain all values of $P_3 P_5$ up to $P_1 P_3/2$. [Of course, P_2 , and hence P_5 , can be on either side of P_3 .] Thus the locus is an arc XP_3Y of the circle with $XP_3 = YP_3$ and $\angle XP_1Y = 2 \tan^{-1} 1/2$. If O is the midpoint of $P_1 P_3$, then O is the center of the circle and $\angle XOY = 4 \tan^{-1} 1/2$ (about 106°).

A5. n people are seated in a circle. A total of nk coins are distributed amongst the people, but not necessarily equally. A *move* is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum number of moves which result in everyone ending up with the same number of coins?

Solution

Label the people from 1 to n , with person i next to person $i+1$, and person n next to person 1. Let person i initially hold c_i coins. Let $d_i = c_i - k$.

It is not obvious how many moves are needed. Clearly *at least* $1/2 \sum |d_i|$ are needed. But one may need more. For example, suppose the starting values of d_i are 0, 1, 0, -1, 0. Then one needs at least 2 moves, not 1.

Obviously $\sum d_i = 0$, so not all d_i can be negative. Relabel if necessary so that $d_1 \geq 0$. Now consider $X = |d_1| + |d_1 + d_2| + |d_1 + d_2 + d_3| + \dots + |d_1 + d_2 + \dots + d_{n-1}|$. Note first that X is zero iff all d_i are zero. Any move between i and $i+1$, *except* one between n and 1, changes X by 1, because only the term $|d_1 + d_2 + \dots + d_i|$ is affected. Thus if we do not make any moves between n and 1, then we need at least X moves to reach the desired final position (with all d_i zero).

Assume $X > 1$. We show how to find a move which reduces X by 1. This requires a little care to avoid specifying a move which might require a person with no coins to transfer one. We are assuming that $d_1 \geq 0$. Take the first i for which $d_{i+1} < 0$. There must be such an i , otherwise all d_i would be non-negative, but they sum to 0, so they would all have to be zero, contradicting $X > 0$. If $d_1 + \dots + d_i > 0$, then we take the move to be a transfer from i to $i+1$. This will reduce $|d_1 + \dots + d_i|$ by 1 and leave the other terms in X unchanged, so it will reduce X by 1. If $d_1 + \dots + d_i$ is not strictly positive, then by the minimality of i we must have $d_1 = d_2 = \dots = d_i = 0$. We know that $d_{i+1} < 0$. Now find the first $j > i+1$ such that $d_j \geq 0$. There must be such a j , otherwise we would have $\sum d_m < 0$. We have $d_1 + \dots + d_{j-1} < 0$, so a transfer from j to $j-1$ will reduce $|d_1 + \dots + d_{j-1}|$ and hence reduce X . Finally note that the move we have chosen leaves $d_1 \geq 0$. Thus we can repeat the process and reduce X to zero in X moves.

We have proved that this procedure minimises the number of moves if we accept the restriction that we do not make any transfers between 1 and n . Thus the full algorithm is: calculate the effect of the transfers from 1 to n and from n to 1 on X . If either of these transfers reduces X by more than 1, then take the move with the larger reduction; otherwise, find a move as above which reduces X by 1; repeat.

10th APMO 1998

A1. S is the set of all possible n -tuples (X_1, X_2, \dots, X_n) where each X_i is a subset of $\{1, 2, \dots, 1998\}$. For each member k of S let $f(k)$ be the number of elements in the union of its n elements. Find the sum of $f(k)$ over all k in S .

Solution

Answer: $1998(2^{1998n} - 2^{1997n})$.

Let $s(n, m)$ be the sum where each X_i is a subset of $\{1, 2, \dots, m\}$. There are 2^m possible X_i and hence 2^{mn} possible n -tuples. We have $s(n, m) = 2^n s(n, m-1) + (2^n - 1)2^{n(m-1)}$ (*). For given any n -tuple $\{X_1, \dots, X_n\}$ of subsets of $\{1, 2, \dots, m-1\}$ we can choose to add m or not (2 choices) to each X_i . So we derive 2^n n -tuples of subsets of $\{1, 2, \dots, m\}$. All but 1 of these have $f(k)$ incremented by 1. The first term in (*) gives the sum for $m-1$ over the increased number of terms and the second term gives the

increments to the $f(k)$ due to the additional element.

Evidently $s(n, 1) = 2^n - 1$. It is now an easy induction to show that $s(n, m) = m(2^{nm} - 2^{n(m-1)})$.

Putting $m = 1998$ we get that the required sum is $1998(2^{1998n} - 2^{1997n})$.

A2. Show that $(36m + n)(m + 36n)$ is not a power of 2 for any positive integers m, n .

Solution

Assume there is a solution. Take $m \leq n$ and the smallest possible m . Now $(36m + n)$ and $(m + 36n)$ must each be powers of 2. Hence 4 divides n and 4 divides m . So $m/2$ and $n/2$ is a smaller solution with $m/2 < m$. Contradiction

A3. Prove that $(1 + x/y)(1 + y/z)(1 + z/x) \geq 2 + 2(x + y + z)/w$ for all positive reals x, y, z , where w is the cube root of xyz .

Solution

$(1 + x/y)(1 + y/z)(1 + z/x) = 1 + x/y + y/x + y/z + z/y + z/x + x/z = (x + y + z)(1/x + 1/y + 1/z) - 1 \geq 3(x + y + z)/w - 1$, by the arithmetic geometric mean inequality, $= 2(x + y + z)/w + (x + y + z)/w - 1 \geq 2(x + y + z) + 3 - 1$, by the arithmetic geometric mean inequality

A4. ABC is a triangle. AD is an altitude. X lies on the circle ABD and Y lies on the circle ACD. X, D and Y are collinear. M is the midpoint of XY and M' is the midpoint of BC. Prove that MM' is perpendicular to AM.

Solution

Take P, Q so that PADB, AQCD are rectangles. Let N be the midpoint of PQ. Then PD is a diameter of the circumcircle of ABC, so PX is perpendicular to XY. Similarly, QY is perpendicular to XY. N is the midpoint of PQ and M' the midpoint of XY, so NM is parallel to PX and hence perpendicular to XY. NADM' is a rectangle, so ND is a diameter of its circumcircle and M must lie on the circumcircle. But AM' is also a diameter, so $\angle AMM' = 90^\circ$.

Thanks to Michael Lipnowski for the above. My original solution is below.

Let P be the circumcenter of ABD and Q the circumcenter of ADC. Let R be the midpoint of AM'. P and Q both lie on the perpendicular bisector of AD, which is parallel to BC and hence also passes through R. We show first that R is the midpoint of PQ.

Let the feet of the perpendiculars from P, Q, R to BC to P', Q', R' respectively. It is sufficient to show that $BP' = BD/2$. $BR' = BM' + M'R' = (BD + DC)/2 + M'D/2 = (BD + DC)/2 + ((BD + DC)/2 - DC)/2 = 3BD/4 + DC/4$, so $P'R' = (BD + DC)/4$. $Q'C = DC/2$, so $BQ' = BD + DC/2$ and $P'Q' = (BD +$

$$DC)/2 = 2P'R'.$$

Now the circumcircle centre P meets XY in X and D, and the circumcircle centre Q meets XY in D and Y. Without loss of generality we may take $XD \geq DY$. Put $XD = 4x$, $DY = 4y$. The circle center R through A, M' and D meets XY in a second point, say M". Let the feet of the perpendiculars from P, Q, R to XY be P", Q", R" respectively. So on XY we have, in order, X, P", M", R", D, Q", Y. Since R is the midpoint of PQ, R" is the midpoint of P"Q". Now P" is the midpoint of XD and Q" is the midpoint of DY, so $P"Q" = XY/2 = 2(x+y)$, so $R"Q" = x+y$. But $DQ" = 2y$, so $R"D = x-y$. R" is the midpoint of M"D, so $M"D = 2(x-y)$ and hence $M"Y = M"D + DY = 2(x-y) + 4y = 2(x+y) = XY/2$. So M" is just M the midpoint of XY. Now AM' is a diameter of the circle center R, so AM is perpendicular to MM'.

A5. What is the largest integer divisible by all positive integers less than its cube root.

Solution

Answer: 420.

Let N be a positive integer satisfying the condition and let n be the largest integer not exceeding its cube root. If $n = 7$, then $3 \cdot 4 \cdot 5 \cdot 7 = 420$ must divide N. But N cannot exceed $8^3 - 1 = 511$, so the largest such N is 420.

If $n \geq 8$, then $3 \cdot 8 \cdot 5 \cdot 7 = 840$ divides N, so $N > 729 = 9^3$. Hence 9 divides N, and hence $3 \cdot 840 = 2520$ divides N. But we show that no $N > 2000$ can satisfy the condition.

Note that $2(x-1)^3 > x^3$ for any $x > 4$. Hence $[x]^3 > x^3/2$ for $x > 4$. So certainly if $N > 2000$, we have $n^3 > N/2$. Now let p_k be the highest power of k which does not exceed n. Then $p_k > n/k$. Hence $p_2 p_3 p_5 > n^3/30 > N/60$. But since $N > 2000$, we have $7 < 11 < n$ and hence $p^2, p^3, p^5, 7, 11$ are all $\leq n$. But $7^2 p^3 p^5 > N$, so N cannot satisfy the condition.

11th APMO 1999

A1. Find the smallest positive integer n such that no arithmetic progression of 1999 reals contains just n integers.

Solution

Answer: 70.

Using a difference of $1/n$, where n does not divide 1999, we can get a progression of 1999 terms with $m = [1998/n]$ or $m = [1998/n] - 1$ integers. Thus $\{0, 1/n, 2/n, \dots, 1998/n\}$ has $m+1$ integers, and $\{1/n, 2/n, \dots, 1999/n\}$ has m integers. So we are ok until n gets so large that the gap between $[1998/n]$ and $[1998/(n+1)]$ is 3 or more. This could happen for $1998/n - 1998/(n+1)$ just over 2 or $n > 31$. So checking, we find $[1998/31] = 64$, $[1998/30] = 66$, $[1998/29] = 68$, $[1998/28] = 71$.

We can get 68 integers with $\{1/29, 2/29, \dots, 1999/29\}$ and 69 with $\{0, 1/29, 2/29, \dots, 1998/29\}$. We

can get 71 with $\{1/28, 2/28, \dots, 1999/28\}$, but we cannot get 70. Note that a progression with irrational difference gives at most 1 integer. A progression with difference a/b , where a and b are coprime integers, gives the same number of integers as a progression with difference $1/b$. So it does not help to widen the class of progressions we are looking at.

A2. The real numbers x_1, x_2, x_3, \dots satisfy $x_{i+j} \leq x_i + x_j$ for all i, j . Prove that $x_1 + x_2/2 + \dots + x_n/n \geq x_n$.

Solution

We use induction. Suppose the result is true for n . We have:

$$x_1 \geq x_1$$

$$x_1 + x_2/2 \geq x_2$$

$$\dots$$

$$x_1 + x_2/2 + \dots + x_n/n \geq x_n$$

Also: $x_1 + 2x_2/2 + \dots + nx_n/n = x_1 + \dots + x_n$

Adding: $(n+1)x_1 + (n+1)x_2/2 + \dots + (n+1)x_n/n \geq 2(x_1 + \dots + x_n)$. But rhs = $(x_1 + x_n) + (x_2 + x_{n-1}) + \dots + (x_n + x_1) \geq n x_{n+1}$. Hence result.

A3. Two circles touch the line AB at A and B and intersect each other at X and Y with X nearer to the line AB . The tangent to the circle AXY at X meets the circle BXY at W . The ray AX meets BW at Z . Show that BW and BX are tangents to the circle XYZ .

Solution

Let angle $ZXW = \alpha$ and angle $ZWX = \beta$. XW is tangent to circle AXY at X , so angle $AYX = \alpha$. AB is tangent to circle AXY at A , so angle $BAX = \alpha$. AB is tangent to circle BXY at B , so angle $ABX = \beta$. Thus, considering triangle ABX , angle $BXZ = \alpha + \beta$. Considering triangle ZXW , angle $BZX = \alpha + \beta$.

$BXYW$ is cyclic, so angle $BYX =$ angle $BWX = \beta$. Hence angle $AYB =$ angle $AYX +$ angle $XYB = \alpha + \beta =$ angle AZB . So $AYZB$ is cyclic. Hence angle $BYZ =$ angle $BAZ = \alpha$. So angle $XYZ =$ angle $XYB +$ angle $BYZ = \alpha + \beta$. Hence angle $BZX =$ angle XYZ , so BZ is tangent to circle XYZ at Z . Similarly angle $BXY =$ angle XYZ , so BX is tangent to circle XYZ at X .

A4. Find all pairs of integers m, n such that $m^2 + 4n$ and $n^2 + 4m$ are both squares.

Solution

Answer: (m, n) or $(n, m) = (0, a^2), (-5, -6), (-4, -4), (a+1, -a)$ where a is a non-negative integer.

Clearly if one of m, n is zero, then the other must be a square and that is a solution.

If both are positive, then $m^2 + 4n$ must be $(m + 2k)^2$ for some positive k , so $n = km + k^2 > m$. But similarly $m > n$. Contradiction. So there are no solutions with m and n positive.

Suppose both are negative. Put $m = -M, n = -N$, so M and N are both positive. Assume $M \geq N$. $M^2 -$

$4N$ is a square, so it must be $(M - 2k)^2$ for some k , so $N = Mk - k^2$. If $M = N$, then $M(k-1) = k^2$, so $k-1$ divides k^2 and hence $k^2 - (k-1)(k+1) = 1$, so $k = 2$ and $M = 4$, giving the solution $(m, n) = (-4, -4)$. So we may assume $M > N$ and hence $M > Mk - k^2 > 0$. But that implies that $k = 1$ or $M-1$ and hence $N = M-1$. [If $M > Mk - k^2$, then $(k-1)M < k^2$. Hence $k = 1$ or $M < k+2$. But $Mk - k^2 > 0$, so $M > k$. Hence $k = 1$ or $M = k+1$.].

But $N^2 - 4M$ is also a square, so $(M-1)^2 - 4M = M^2 - 6M + 1$ is a square. But $(M-3)^2 > M^2 - 6M + 1$ and $(M-4)^2 < M^2 - 6M + 1$ for $M \geq 8$, so the only possible solutions are $M = 1, 2, \dots, 7$. Checking, we find that only $M = 6$ gives $M^2 - 6M + 1$ a square. This gives the solution $(m, n) = (-6, -5)$. Obviously, there is also the solution $(-5, -6)$.

Finally, consider the case of opposite signs. Suppose $m = M > 0$, $n = -N < 0$. Then $N^2 + 4M$ is a square, so by the argument above $M > N$. But $M^2 - 4N$ is a square and so the argument above gives $N = M-1$. Now we can easily check that $(m, n) = (M, -(M-1))$ is a solution for any positive M .

A5. A set of $2n+1$ points in the plane has no three collinear and no four concyclic. A circle is said to divide the set if it passes through 3 of the points and has exactly $n - 1$ points inside it. Show that the number of circles which divide the set is even iff n is even.

Solution

Take two of the points, A and B , and consider the $2n-1$ circles through A and B . We will show that the number of these circles which divide the set is odd. The result then follows almost immediately, because the number of pairs A, B is $(2n+1)2n/2 = N$, say. The total number of circles which divide the set is a sum of N odd numbers divided by 3 (because each such circle will be counted three times). If n is even, then N is even, so a sum of N odd numbers is even. If n is odd, then N is odd, so a sum of N odd numbers is odd. Dividing by 3 does not change the parity.

Their centers all lie on the perpendicular bisector of AB . Label them $C_1, C_2, \dots, C_{2n-1}$, where the center of C_i lies to the left of C_j on the bisector iff $i < j$. We call the two half-planes created by AB the left-hand half-plane L and the right-hand half-plane R correspondingly. Let the third point of the set on C_i be X_i . Suppose $i < j$. Then C_i contains all points of C_j that lie in L and C_j contains all points of C_i that lie in R . So X_i lies inside C_j iff X_i lies in R and X_j lies inside C_i iff X_j lies in L .

Now plot $f(i)$, the number of points in the set that lie inside C_i , as a function of i . If X_i and X_{i+1} are on opposite sides of AB , then $f(i+1) = f(i)$. If they are both in L , then $f(i+1) = f(i) - 1$, and if they are both in R , then $f(i+1) = f(i) + 1$. Suppose m of the X_i lie in L and $2n-1-m$ lie in R . Now suppose $f(i) = n-2$, $f(i+1) = f(i+2) = \dots = f(i+j) = n-1$, $f(i+j+1) = n$. Then j must be odd. For X_i and X_{i+1} must lie in R . Then the points must alternate, so X_{i+2} lies in L , X_{i+3} lies in R etc. But X_{i+j} and X_{i+j+1} must lie in R . Hence j must be odd. On the other hand, if $f(i+j+1) = n-2$, then j must be even. So the parity of the number of C_1, C_2, \dots, C_i which divide the set only changes when f crosses the line $n-1$ from one side to the other. We now want to say that f starts on one side of the line $n-1$ and ends on the other, so the final parity must be odd. Suppose there are m points in L and hence $2n-1-m$ in R . Without loss of generality we may take $m \leq n-1$. The first circle C_1 contains all the points in L except X_1 if it is in L . So $f(1) = m$ or $m-1$. Similarly the last circle C_{2n-1} contains all the points in R except X_{2n-1} if it is in R . So $f(2n-1) = 2n-1-m$ or $2n-2-m$. Hence if $m < n-1$, then $f(1) = m$ or $m-1$, so $f(1) < n-1$. But $2n-1-m \geq$

$n+1$, so $f(2n-1) > n-1$. So in this case we are done.

However, there are complications if $m = n-1$. We have to consider 4 cases. Case (1): $m = n-1$, X_1 lies in R, X_{2n-1} lies in L. Hence $f(1) = n-1$, $f(2n-1) = n > n-1$. So f starts on the line $n-1$. If it first leaves it downwards, then for the previous point i , X_i is in L and hence there were an even number of points up to i on the line. So the parity is the same as if $f(1)$ was below the line. $f(2n-1)$ is above the line, so we get an odd number of points on the line. If f first leaves the line upwards, then for the previous point i , X_i is in R and hence there were an odd number of points up to i on the line. So again the parity is the same as if $f(1)$ was below the line.

Case (2): $m = n-1$, X_1 lies in R, X_{2n-1} lies in R. Hence $f(1) = f(2n-1) = n-1$. As in case (1) the parity is the same as if $f(1)$ was below the line. If the last point j with $f(j)$ not on the line has $f(j) < n-1$, then (since X_{2n-1} lies in R) there are an odd number of points above j , so the parity is the same as if $f(2n-1)$ was above the line. Similarly if $f(j) > n-1$, then there are an even number of points above j , so again the parity is the same as if $f(2n-1)$ was above the line.

Case (3): $m = n-1$, X_1 lies in L, X_{2n-1} lies in L. Hence $f(1) = n-2$, $f(2n-1) = n$. So case has already been covered.

Case (4): $m = n-1$, X_1 lies in L, X_{2n-1} lies in R. So $f(1) = n-2$, $f(2n-1) = n-1$. As in case (2), the parity is the same as if $f(2n-1)$ was above the line.

12th APMO 2000

A1. Find $a_1^3/(1 - 3a_1 + 3a_1^2) + a_2^3/(1 - 3a_2 + 3a_2^2) + \dots + a_{101}^3/(1 - 3a_{101} + 3a_{101}^2)$, where $a_n = n/101$.

Solution

Answer: 51.

The n th term is $a_n^3/(1 - 3a_n + 3a_n^2) = a_n^3/((1 - a_n)^3 + a_n^3) = n^3/((101 - n)^3 + n^3)$. Hence the sum of the n th and $(101-n)$ th terms is 1. Thus the sum from $n = 1$ to 100 is 50. The last term is 1, so the total sum is 51.

A2. Find all permutations a_1, a_2, \dots, a_9 of $1, 2, \dots, 9$ such that $a_1 + a_2 + a_3 + a_4 = a_4 + a_5 + a_6 + a_7 = a_7 + a_8 + a_9 + a_1$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2 = a_7^2 + a_8^2 + a_9^2 + a_1^2$.

Solution

We may start by assuming that $a_1 < a_4 < a_7$ and that $a_2 < a_3, a_5 < a_6, a_8 < a_9$.

Note that $1 + \dots + 9 = 45$ and $1^2 + \dots + 9^2 = 285$. Adding the three square equations together we get $(a_1^2 + \dots + a_9^2) + a_1^2 + a_4^2 + a_7^2 = 285 + a_1^2 + a_4^2 + a_7^2$. The total must be a multiple of 3. But 285 is a

multiple of 3, so $a_1^2 + a_4^2 + a_7^2$ must be a multiple of 3. Now $3^2, 6^2$ and 9^2 are all congruent to 0 mod 3 and the other squares are all congruent to 1 mod 3. Hence either a_1, a_4 and a_7 are all multiples of 3, or none of them are. Since 45 is also a multiple of three a similar argument with the three linear equations shows that $a_1 + a_4 + a_7$ is a multiple of 3. So if none of a_1, a_4, a_7 are multiples of 3, then they are all congruent to 1 mod 3 or all congruent to 2 mod 3. Thus we have three cases: (1) $a_1 = 3, a_4 = 6, a_7 = 9$, (2) $a_1 = 1, a_4 = 4, a_7 = 7$, and (3) $a_1 = 2, a_4 = 5, a_7 = 8$.

In case (1), we have that each sum of squares equals 137. Hence $a_8^2 + a_9^2 = 47$. But 47 is not a sum of two squares, so this case gives no solutions.

In case (2), we have that each sum of squares is 117. Hence $a_5^2 + a_6^2 = 52$. But the only way of writing 52 as a sum of two squares is $4^2 + 6^2$ and 4 is already taken by a_4 , so this case gives no solutions.

In case (3), we have that each sum of squares is 126 and each linear sum 20. We quickly find that the only solution is 2, 4, 9, 5, 1, 6, 8, 3, 7.

Obviously, this generates a large number of equivalent solutions. We can interchange a_2 and a_3 , or a_5 and a_6 , or a_8 and a_9 . We can also permute a_1, a_4 and a_7 . So we get a total of $2 \times 2 \times 2 \times 6 = 48$ solutions.

A3. ABC is a triangle. The angle bisector at A meets the side BC at X. The perpendicular to AX at X meets AB at Y. The perpendicular to AB at Y meets the ray AX at R. XY meets the median from A at S. Prove that RS is perpendicular to BC.

Solution

Let the line through C parallel to AX meet the ray BA at C'. Let the perpendicular from B meet the ray C'C at T and the ray AX at U. Let the line from C parallel to BT meet BA at V and let the perpendicular from V meet BT at W. So CVWT is a rectangle.

AU bisects $\angle CAV$ and CV is perpendicular to AU, so U is the midpoint of WT. Hence the intersection N of AU and CW is the center of the rectangle and, in particular, the midpoint of CW. Let M be the midpoint of BC. Then since M, N are the midpoints of the sides CB and CW of the triangle CBW, $MN = BW/2$.

Since CC' is parallel to AX, $\angle CC'A = \angle BAX = \angle CAX = \angle C'CA$, so $AC' = AC$. Let A' be the midpoint of CC'. Then $AU = C'T - C'A'$. But N is the center of the rectangle CTWV, so $NU = CT/2$ and $AN = AU - NU = C'T - C'A' - CT/2 = C'T/2$. Hence $MN/AN = BW/C'T$. But MN is parallel to BW and XY, so $SX/AX = MN/AN = BW/C'T$.

Now AX is parallel to VW and XY is parallel to BW, so AXY and VWB are similar and $AX/XY = VW/BW = CT/BW$. Hence $SX/XY = (SX/AX) (AX/XY) = CT/C'T$.

YX is an altitude of the right-angled triangle AXR, so AXY and YXR are similar. Hence $XY/XR = XA/XY$. But AXY and C'TB are similar, so $XA/XY = C'T/BT$. Hence $SX/XR = (SX/XY) (XY/XR) = (CT/C'T) (C'T/BT) = CT/BT$. But angles CTB and SXR are both right angles, so SXR and CTB are similar. But XR is perpendicular to BT, so SR is perpendicular to BC.

A4. If $m < n$ are positive integers prove that $n^m / (m^m (n-m)^{n-m}) > n! / (m! (n-m)!) > n^m / (m^{m(n+1)} (n-m)^{n-m})$.

Solution

The key is to consider the binomial expansion $(m + n-m)^n$. This is a sum of positive terms, one of which is $nCm m^m (n-m)^{n-m}$, where nCm is the binomial coefficient $n! / (m! (n-m)!)$. Hence $nCm m^m (n-m)^{n-m} < n^n$, which is one of the required inequalities.

We will show that $nCm m^m (n-m)^{n-m}$ is the largest term in the binomial expansion. It then follows that $(n+1) nCm m^m (n-m)^{n-m} > n^n$, which is the other required inequality.

Comparing the r th term $nCr m^r (n-m)^{n-r}$ with the $r+1$ th term $nCr+1 m^{r+1} (n-m)^{n-r-1}$ we see that the r th term is strictly larger for $r \geq m$ and smaller for $r < m$. Hence the m th term is larger than the succeeding terms and also larger than the preceding terms.

A5. Given a permutation s_0, s_2, \dots, s_n of $0, 1, 2, \dots, n$, we may *transform* it if we can find i, j such that $s_i = 0$ and $s_j = s_{i-1} + 1$. The new permutation is obtained by transposing s_i and s_j . For which n can we obtain $(1, 2, \dots, n, 0)$ by repeated transformations starting with $(1, n, n-1, \dots, 3, 2, 0)$?

Solution

Experimentation shows that we can do it for $n=1$ (already there), $n = 2$ (already there), $3, 7, 15$, but not for $n = 4, 5, 6, 8, 9, 10, 11, 12, 13, 14$. So we conjecture that it is possible just for $n = 2^m - 1$ and for $n = 2$.

Notice that there is at most one transformation possible. If $n = 2m$, then we find that after $m-1$ transformations we reach

1 n 0 n-2 n-1 n-4 n-3 ... 4 5 2 3
and we can go no further. So n even fails for $n > 2$.

If $n = 15$ we get successively:

1	15	14	13	12	11	10	9	8	7	6	5	4	3	2	0	start
1	0	14	15	12	13	10	11	8	9	6	7	4	5	2	3	after 7 moves
1	2	3	0	12	13	14	15	8	9	10	11	4	5	6	7	after 8 more moves
1	2	3	4	5	6	7	0	8	9	10	11	12	13	14	15	after 8 more moves
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0	after 8 more moves

This pattern is general. Suppose $n = 2^m - 1$. Let P_0 be the starting position and P_r be the position:
1 2 3 ... R-1 0, n-R+1 n-R+2 n-R+3 ... n, n-2R+1 n-2R+2 ... n-R, ...
, R R+1 ... 2R-1

Here R denotes 2^r and the commas highlight that, after the initial $1 2 \dots R-1 0$, we have increasing runs of R terms. If we start from P_r , then the 0 is transposed successively with $R, 3R, 5R, \dots, n-R+1$, then with $R+1, 3R+1, \dots, n-R+2$, and so on up to $2R-1, 4R-1, \dots, n$. But that gives P_{r+1} . It is also easy to check that P_0 leads to P_1 and that P_m is the required finishing position. Thus the case $n = 2^m - 1$

works.

Now suppose n is odd but not of the form $2^m - 1$. Then we can write $n = (2a + 1)2^b - 1$ (just take 2^b as the highest power of 2 dividing $n + 1$). We can now define P_0, P_1, \dots, P_b as before. As before we will reach P_b :

1 2 $\frac{1}{2}$ $B-1$ 0, $2aB$ $2aB+1$ $\frac{1}{2}$ $(2a+1)B-1$, $(2a-1)B$ $\frac{1}{2}$ $2aB-1$, $\frac{1}{2}$, $3B$, $3B+1$, $\frac{1}{2}$ $4B-1$, $2B$, $2B+1$, $\frac{1}{2}$, $3B-1$, B , $B+1$, $\frac{1}{2}$, $2B-1$

where $B = 2^b - 1$. But then the 0 is transposed successively with $B, 3B, 5B, \dots, (2a-1)B$, which puts it immediately to the right of $(2a+1)B-1 = n$, so no further transformations are possible and $n = (2a+1)2^b - 1$ fails

XIII Asian Pacific Mathematics Olympiad

March, 2001

Time allowed: 4 hours

No calculators to be used

Each question is worth 7 points

Problem 1.

For a positive integer n let $S(n)$ be the sum of digits in the decimal representation of n . Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n . Let $T(n)$ be the sum of all stumps of n . Prove that $n = S(n) + 9T(n)$.

Problem 2.

Find the largest positive integer N so that the number of integers in the set $\{1, 2, \dots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Problem 3.

Let two equal regular n -gons S and T be located in the plane such that their intersection is a $2n$ -gon ($n \geq 3$). The sides of the polygon S are coloured in red and the sides of T in blue.

Prove that the sum of the lengths of the blue sides of the polygon $S \cap T$ is equal to the sum of the lengths of its red sides.

Problem 4.

A point in the plane with a cartesian coordinate system is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

Problem 5.

Find the greatest integer n , such that there are $n+4$ points $A, B, C, D, X_1, \dots, X_n$ in the plane with $AB \neq CD$ that satisfy the following condition: for each $i = 1, 2, \dots, n$ triangles ABX_i and CDX_i are equal.

XIV Asian Pacific Mathematics Olympiad March 2002

Time allowed: 4 hours
No calculators are to be used
Each question is worth 7 points

Problem 1.

Let $a_1, a_2, a_3, \dots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \dots + a_n}{n}.$$

Prove that

$$a_1! a_2! \dots a_n! \geq ([A_n]!)^n,$$

where $[A_n]$ is the greatest integer less than or equal to A_n , and $a! = 1 \times 2 \times \dots \times a$ for $a \geq 1$ (and $0! = 1$).
When does equality hold?

Problem 2.

Find all positive integers a and b such that

$$\frac{a^2 + b}{b^2 - a} \quad \text{and} \quad \frac{b^2 + a}{a^2 - b}$$

are both integers.

Problem 3.

Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ . Let T be the point common to the segments BP and CQ . Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Problem 4.

Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Problem 5.

Let \mathbf{R} denote the set of all real numbers. Find all functions f from \mathbf{R} to \mathbf{R} satisfying:

- (i) there are only finitely many s in \mathbf{R} such that $f(s) = 0$, and
- (ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in \mathbf{R} .

XV Asian Pacific Mathematics Olympiad March 2003

Time allowed: 4 hours

No calculators are to be used

Each question is worth 7 points

Problem 1.

Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \dots, 8$. Determine all possible values of f .

Problem 2.

Suppose $ABCD$ is a square piece of cardboard with side length a . On a plane are two parallel lines ℓ_1 and ℓ_2 , which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Problem 3.

Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k . You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:

- (a) if $n = 2p_k$, then n does not divide $(n - k)!$;
- (b) if $n > 2p_k$, then n divides $(n - k)!$.

Problem 4.

Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

Problem 5.

Given two positive integers m and n , find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

XVI Asian Pacific Mathematics Olympiad March 2004

Time allowed: 4 hours

No calculators are to be used

Each question is worth 7 points

Problem 1.

Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \quad \text{is an element of } S \text{ for all } i, j \text{ in } S,$$

where (i, j) is the greatest common divisor of i and j .

Problem 2.

Let O be the circumcentre and H the orthocentre of an acute triangle ABC . Prove that the area of one of the triangles AOH , BOH and COH is equal to the sum of the areas of the other two.

Problem 3.

Let a set S of 2004 points in the plane be given, no three of which are collinear. Let \mathcal{L} denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S , the number of lines in \mathcal{L} which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ .

Problem 4.

For a real number x , let $[x]$ stand for the largest integer that is less than or equal to x . Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n .

Problem 5.

Prove that

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)$$

for all real numbers $a, b, c > 0$.



XVII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours

Each problem is worth 7 points

** The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Prove that for every irrational real number a , there are irrational real numbers b and b' so that $a + b$ and ab' are both rational while ab and $a + b'$ are both irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}.$$

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t + 1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t . Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i - k| + |j - \ell| = 1$.

Problem 5. In a triangle ABC , points M and N are on sides AB and AC , respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC , respectively. Express the ratio MN/BC in terms of R and r .

XVII APMO - March, 2005

Problems and Solutions

Problem 1. Prove that for every irrational real number a , there are irrational real numbers b and b' so that $a + b$ and ab' are both rational while ab and $a + b'$ are both irrational.

(Solution) Let a be an irrational number. If a^2 is irrational, we let $b = -a$. Then, $a + b = 0$ is rational and $ab = -a^2$ is irrational. If a^2 is rational, we let $b = a^2 - a$. Then, $a + b = a^2$ is rational and $ab = a^2(a - 1)$. Since

$$a = \frac{ab}{a^2} + 1$$

is irrational, so is ab .

Now, we let $b' = \frac{1}{a}$ or $b' = \frac{2}{a}$. Then $ab' = 1$ or 2 , which is rational. Note that

$$a + b' = \frac{a^2 + 1}{a} \quad \text{or} \quad a + b' = \frac{a^2 + 2}{a}.$$

Since,

$$\frac{a^2 + 2}{a} - \frac{a^2 + 1}{a} = \frac{1}{a},$$

at least one of them is irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}.$$

(Solution) Observe that

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}. \quad (1)$$

In fact, this is equivalent to $(2+x^2)^2 \geq 4(1+x^3)$, or $x^2(x-2)^2 \geq 0$. Notice that equality holds in (1) if and only if $x = 2$.

We substitute x by a, b, c in (1), respectively, to find

$$\begin{aligned} & \frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \\ & \geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+a^2)}. \end{aligned} \quad (2)$$

We combine the terms on the right hand side of (2) to obtain

$$\text{Left hand side of (2)} \geq \frac{2S(a, b, c)}{36 + S(a, b, c)} = \frac{2}{1 + 36/S(a, b, c)}, \quad (3)$$

where $S(a, b, c) := 2(a^2 + b^2 + c^2) + (ab)^2 + (bc)^2 + (ca)^2$. By AM-GM inequality, we have

$$\begin{aligned} a^2 + b^2 + c^2 & \geq 3\sqrt[3]{(abc)^2} = 12, \\ (ab)^2 + (bc)^2 + (ca)^2 & \geq 3\sqrt[3]{(abc)^4} = 48. \end{aligned}$$

Note that the equalities holds if and only if $a = b = c = 2$. The above inequalities yield

$$S(a, b, c) = 2(a^2 + b^2 + c^2) + (ab)^2 + (bc)^2 + (ca)^2 \geq 72. \quad (4)$$

Therefore

$$\frac{2}{1 + 36/S(a, b, c)} \geq \frac{2}{1 + 36/72} = \frac{4}{3}, \quad (5)$$

which is the required inequality.

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

(Solution) Suppose that one side of a triangle has length n . Then it can be cut into n^2 congruent triangles which are similar to the original one and whose corresponding sides to the side of length n have lengths 1.

Since $2005 = 5 \times 401$ where 5 and 401 are primes and both primes are of the type $4k + 1$, it is representable as a sum of two integer squares. Indeed, it is easy to see that

$$\begin{aligned} 2005 &= 5 \times 401 = (2^2 + 1)(20^2 + 1) \\ &= 40^2 + 20^2 + 2^2 + 1 \\ &= (40 - 1)^2 + 2 \times 40 + 20^2 + 2^2 \\ &= 39^2 + 22^2. \end{aligned}$$

Let ABC be a right-angled triangle with the legs AB and BC having lengths 39 and 22, respectively. We draw the altitude BK , which divides ABC into two similar triangles. Now we divide ABK into 39^2 congruent triangles as described above and BCK into 22^2 congruent triangles. Since ABK is similar to BKC , all 2005 triangles will be congruent.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t + 1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t . Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i - k| + |j - \ell| = 1$.

(Solution) At most $n^2 + c^2 - nc - c$ houses can be saved. This can be achieved under the following order of defending:

$$(2, c), (2, c + 1); (3, c - 1), (3, c + 2); (4, c - 2), (4, c + 3); \dots \\ (c + 1, 1), (c + 1, 2c); (c + 1, 2c + 1), \dots, (c + 1, n). \quad (6)$$

Under this strategy, there are

- 2 columns (column numbers $c, c + 1$) at which $n - 1$ houses are saved
- 2 columns (column numbers $c - 1, c + 2$) at which $n - 2$ houses are saved
- ...
- 2 columns (column numbers $1, 2c$) at which $n - c$ houses are saved
- $n - 2c$ columns (column numbers $n - 2c + 1, \dots, n$) at which $n - c$ houses are saved

Adding all these we obtain:

$$2[(n - 1) + (n - 2) + \dots + (n - c)] + (n - 2c)(n - c) = n^2 + c^2 - cn - c. \quad (7)$$

We say that a house indexed by (i, j) is at level t if $|i - 1| + |j - c| = t$. Let $d(t)$ be the number of houses at level t defended by time t , and $p(t)$ be the number of houses at levels greater than t defended by time t . It is clear that

$$p(t) + \sum_{i=1}^t d(i) \leq t \text{ and } p(t + 1) + d(t + 1) \leq p(t) + 1.$$

Let $s(t)$ be the number of houses at level t which are not burning at time t . We prove that

$$s(t) \leq t - p(t) \leq t$$

for $1 \leq t \leq n - 1$ by induction. It is obvious when $t = 1$. Assume that it is true for $t = k$. The union of the neighbors of any $k - p(k) + 1$ houses at level $k + 1$ contains at least $k - p(k) + 1$ vertices at level k . Since $s(k) \leq k - p(k)$, one of these houses at level k is burning. Therefore, at most $k - p(k)$ houses at level $k + 1$ have no neighbor burning. Hence we have

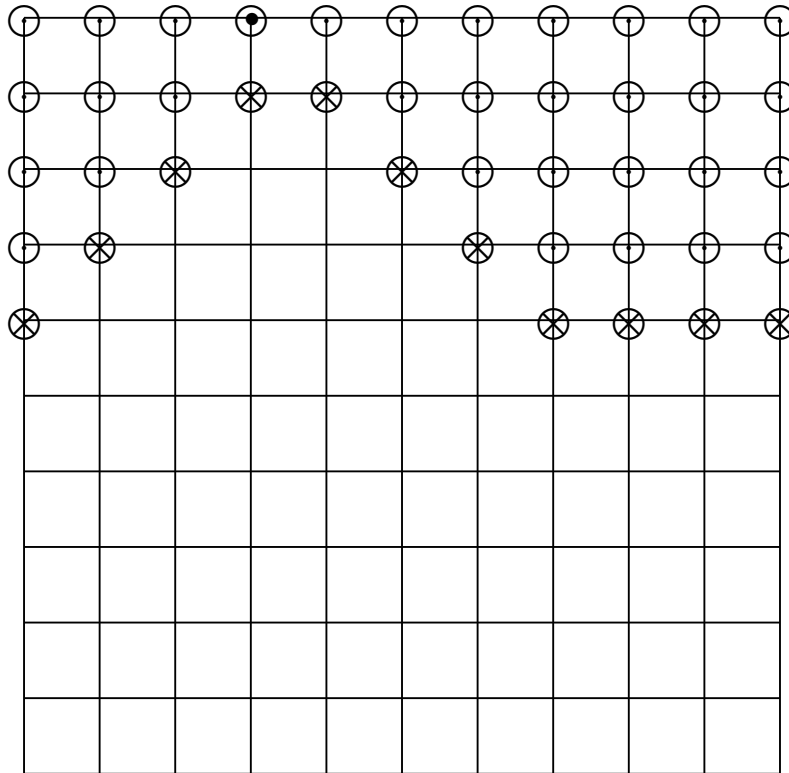
$$\begin{aligned} s(k + 1) &\leq k - p(k) + d(k + 1) \\ &= (k + 1) - (p(k) + 1 - d(k + 1)) \\ &\leq (k + 1) - p(k + 1). \end{aligned}$$

We now prove that the strategy given above is optimal. Since

$$\sum_{t=1}^{n-1} s(t) \leq \binom{n}{2},$$

the maximum number of houses at levels less than or equal to $n - 1$, that can be saved under any strategy is at most $\binom{n}{2}$, which is realized by the strategy above. Moreover, at levels bigger than $n - 1$, every house is saved under the strategy above.

The following is an example when $n = 11$ and $c = 4$. The houses with \circ mark are burned. The houses with \otimes mark are blocked ones and hence those and the houses below them are saved.



Problem 5. In a triangle ABC , points M and N are on sides AB and AC , respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC , respectively. Express the ratio MN/BC in terms of R and r .

(Solution) Let ω , O and I be the circumcircle, the circumcenter and the incenter of ABC , respectively. Let D be the point of intersection of the line BI and the circle ω such that $D \neq B$. Then D is the midpoint of the arc AC . Hence $OD \perp CN$ and $OD = R$.

We first show that triangles MNC and IOD are similar. Because $BC = BM$, the line BI (the bisector of $\angle MBC$) is perpendicular to the line CM . Because $OD \perp CN$ and $ID \perp MC$, it follows that

$$\angle ODI = \angle NCM \quad (8)$$

Let $\angle ABC = 2\beta$. In the triangle BCM , we have

$$\frac{CM}{NC} = \frac{CM}{BC} = 2 \sin \beta \quad (9)$$

Since $\angle DIC = \angle DCI$, we have $ID = CD = AD$. Let E be the point of intersection of the line DO and the circle ω such that $E \neq D$. Then DE is a diameter of ω and $\angle DEC = \angle DBC = \beta$. Thus we have

$$\frac{DI}{OD} = \frac{CD}{OD} = \frac{2R \sin \beta}{R} = 2 \sin \beta. \quad (10)$$

Combining equations (8), (9), and (10) shows that triangles MNC and IOD are similar. It follows that

$$\frac{MN}{BC} = \frac{MN}{NC} = \frac{IO}{OD} = \frac{IO}{R}. \quad (11)$$

The well-known Euler's formula states that

$$OI^2 = R^2 - 2Rr. \quad (12)$$

Therefore,

$$\frac{MN}{BC} = \sqrt{1 - \frac{2r}{R}}. \quad (13)$$

(Alternative Solution) Let a (resp., b , c) be the length of BC (resp., AC , AB). Let α (resp., β , γ) denote the angle $\angle BAC$ (resp., $\angle ABC$, $\angle ACB$). By introducing coordinates $B = (0, 0)$, $C = (a, 0)$, it is immediate that the coordinates of M and N are

$$M = (a \cos \beta, a \sin \beta), \quad N = (a - a \cos \gamma, a \sin \gamma), \quad (14)$$

respectively. Therefore,

$$\begin{aligned}
(MN/BC)^2 &= [(a - a \cos \gamma - a \cos \beta)^2 + (a \sin \gamma - a \sin \beta)^2]/a^2 \\
&= (1 - \cos \gamma - \cos \beta)^2 + (\sin \gamma - \sin \beta)^2 \\
&= 3 - 2 \cos \gamma - 2 \cos \beta + 2(\cos \gamma \cos \beta - \sin \gamma \sin \beta) \\
&= 3 - 2 \cos \gamma - 2 \cos \beta + 2 \cos(\gamma + \beta) \\
&= 3 - 2 \cos \gamma - 2 \cos \beta - 2 \cos \alpha \\
&= 3 - 2(\cos \gamma + \cos \beta + \cos \alpha).
\end{aligned} \tag{15}$$

Now we claim

$$\cos \gamma + \cos \beta + \cos \alpha = \frac{r}{R} + 1. \tag{16}$$

From

$$\begin{aligned}
a &= b \cos \gamma + c \cos \beta \\
b &= c \cos \alpha + a \cos \gamma \\
c &= a \cos \beta + b \cos \alpha
\end{aligned} \tag{17}$$

we get

$$a(1 + \cos \alpha) + b(1 + \cos \beta) + c(1 + \cos \gamma) = (a + b + c)(\cos \alpha + \cos \beta + \cos \gamma). \tag{18}$$

Thus

$$\begin{aligned}
&\cos \alpha + \cos \beta + \cos \gamma \\
&= \frac{1}{a + b + c} (a(1 + \cos \alpha) + b(1 + \cos \beta) + c(1 + \cos \gamma)) \\
&= \frac{1}{a + b + c} \left(a \left(1 + \frac{b^2 + c^2 - a^2}{2bc} \right) + b \left(1 + \frac{a^2 + c^2 - b^2}{2ac} \right) + c \left(1 + \frac{a^2 + b^2 - c^2}{2ab} \right) \right) \\
&= \frac{1}{a + b + c} \left(a + b + c + \frac{a^2(b^2 + c^2 - a^2) + b^2(a^2 + c^2 - b^2) + c^2(a^2 + b^2 - c^2)}{2abc} \right) \\
&= 1 + \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2abc(a + b + c)}.
\end{aligned} \tag{19}$$

On the other hand, from $R = \frac{a}{2 \sin \alpha}$ it follows that

$$\begin{aligned}
R^2 &= \frac{a^2}{4(1 - \cos^2 \alpha)} = \frac{a^2}{4 \left(1 - \left(\frac{b^2 + c^2 - a^2}{2bc} \right)^2 \right)} \\
&= \frac{a^2b^2c^2}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}.
\end{aligned} \tag{20}$$

Also from $\frac{1}{2}(a+b+c)r = \frac{1}{2}bc \sin \alpha$, it follows that

$$\begin{aligned} r^2 &= \frac{b^2c^2(1 - \cos^2 \alpha)}{(a+b+c)^2} = \frac{b^2c^2 \left(1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right)}{(a+b+c)^2} \\ &= \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{4(a+b+c)^2}. \end{aligned} \quad (21)$$

Combining (19), (20) and (21), we get (16) as desired.

Finally, by (15) and (16) we have

$$\frac{MN}{BC} = \sqrt{1 - \frac{2r}{R}}. \quad (22)$$

Another proof of (16) from R.A. Johnson's "Advanced Euclidean Geometry"¹:

Construct the perpendicular bisectors OD, OE, OF , where D, E, F are the midpoints of BC, CA, AB , respectively. By Ptolemy's Theorem applied to the cyclic quadrilateral $OEOF$, we get

$$\frac{a}{2} \cdot R = \frac{b}{2} \cdot OF + \frac{c}{2} \cdot OE.$$

Similarly

$$\frac{b}{2} \cdot R = \frac{c}{2} \cdot OD + \frac{a}{2} \cdot OF, \quad \frac{c}{2} \cdot R = \frac{a}{2} \cdot OE + \frac{b}{2} \cdot OD.$$

Adding, we get

$$sR = OD \cdot \frac{b+c}{2} + OE \cdot \frac{c+a}{2} + OF \cdot \frac{a+b}{2}, \quad (23)$$

where s is the semiperimeter. But also, the area of triangle OBC is $OD \cdot \frac{a}{2}$, and adding similar formulas for the areas of triangles OCA and OAB gives

$$rs = \triangle ABC = OD \cdot \frac{a}{2} + OE \cdot \frac{b}{2} + OF \cdot \frac{c}{2} \quad (24)$$

Adding (23) and (24) gives $s(R+r) = s(OD+OE+OF)$, or

$$OD + OE + OF = R + r.$$

Since $OD = R \cos A$ etc., (16) follows.

¹This proof was introduced to the coordinating country by Professor Bill Sands of Canada.



XVIII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours

Each problem is worth 7 points

** The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever a_1, a_2, \dots, a_n are real numbers such that $a_1 + a_2 + \dots + a_n$ is an integer, there exists some i such that $|a_i - \frac{1}{2}| \geq f(n)$.

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i , where i is an integer (not necessarily positive).

Problem 3. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5 . Here, we assume that all the checkers are identical.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB . Let O_1 be the circle tangent to the line AB at P and tangent to the circle O . Let ℓ be the tangent line, different from the line AB , to O_1 passing through A . Let C be the intersection point, different from A , of ℓ and O . Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC . Prove that the circle O_2 is tangent to the circle O .

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever a_1, a_2, \dots, a_n are real numbers such that $a_1 + a_2 + \dots + a_n$ is an integer, there exists some i such that $|a_i - \frac{1}{2}| \geq f(n)$.

(Solution) The answer is

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

First, assume that n is even. If $a_i = \frac{1}{2}$ for all i , then the sum $a_1 + a_2 + \dots + a_n$ is an integer. Since $|a_i - \frac{1}{2}| = 0$ for all i , we may conclude $f(n) = 0$ for any even n .

Now assume that n is odd. Suppose that $|a_i - \frac{1}{2}| < \frac{1}{2n}$ for all $1 \leq i \leq n$. Then, since $\sum_{i=1}^n a_i$ is an integer,

$$\frac{1}{2} \leq \left| \sum_{i=1}^n a_i - \frac{n}{2} \right| \leq \sum_{i=1}^n \left| a_i - \frac{1}{2} \right| < \frac{1}{2n} \cdot n = \frac{1}{2},$$

a contradiction. Thus $|a_i - \frac{1}{2}| \geq \frac{1}{2n}$ for some i , as required. On the other hand, putting $n = 2m + 1$ and $a_i = \frac{m}{2m+1}$ for all i gives $\sum a_i = m$, while

$$\left| a_i - \frac{1}{2} \right| = \frac{1}{2} - \frac{m}{2m+1} = \frac{1}{2(2m+1)} = \frac{1}{2n}$$

for all i . Therefore, $f(n) = \frac{1}{2n}$ is the best possible for any odd n .

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i , where i is an integer (not necessarily positive).

(Solution) We will prove this statement by induction using the equality

$$\tau^2 = \tau + 1.$$

If $n = 1$, then $1 = \tau^0$. Suppose that $n - 1$ can be written as a finite sum of integral powers of τ , say

$$n - 1 = \sum_{i=-k}^k a_i \tau^i, \tag{1}$$

where $a_i \in \{0, 1\}$ and $n \geq 2$. We will write (1) as

$$n - 1 = a_k \cdots a_1 a_0 \cdot a_{-1} a_{-2} \cdots a_{-k}. \tag{2}$$

For example,

$$1 = 1.0 = 0.11 = 0.1011 = 0.101011.$$

Firstly, we will prove that we may assume that in (2) we have $a_i a_{i+1} = 0$ for all i with $-k \leq i \leq k-1$. Indeed, if we have several occurrences of 11, then we take the leftmost such occurrence. Since we may assume that it is preceded by a 0, we can replace 011 with 100 using the identity $\tau^{i+1} + \tau^i = \tau^{i+2}$. By doing so repeatedly, if necessary, we will eliminate all occurrences of two 1's standing together. Now we have the representation

$$n - 1 = \sum_{i=-K}^K b_i \tau^i, \quad (3)$$

where $b_i \in \{0, 1\}$ and $b_i b_{i+1} = 0$.

If $b_0 = 0$ in (3), then we just add $1 = \tau^0$ to both sides of (3) and we are done.

Suppose now that there is 1 in the unit position of (3), that is $b_0 = 1$. If there are two 0's to the right of it, i.e.

$$n - 1 = \dots 1.00\dots,$$

then we can replace 1.00 with 0.11 because $1 = \tau^{-1} + \tau^{-2}$, and we are done because we obtain 0 in the unit position. Thus we may assume that

$$n - 1 = \dots 1.010\dots.$$

Again, if we have $n - 1 = \dots 1.0100\dots$, we may rewrite it as

$$n - 1 = \dots 1.0100\dots = \dots 1.0011\dots = \dots 0.1111\dots$$

and obtain 0 in the unit position. Therefore, we may assume that

$$n - 1 = \dots 1.01010\dots.$$

Since the number of 1's is finite, eventually we will obtain an occurrence of 100 at the end, i.e.

$$n - 1 = \dots 1.01010\dots 100.$$

Then we can shift all 1's to the right to obtain 0 in the unit position, i.e.

$$n - 1 = \dots 0.11\dots 11,$$

and we are done.

Problem 3. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5 . Here, we assume that all the checkers are identical.

(Solution) Note that $r = \binom{p^2}{p} - p$. Hence, it suffices to show that

$$(p^2 - 1)(p^2 - 2) \cdots (p^2 - (p - 1)) - (p - 1)! \equiv 0 \pmod{p^4}. \quad (1)$$

Now, let

$$f(x) := (x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} + s_{p-2}x^{p-2} + \cdots + s_1x + s_0. \quad (2)$$

Then the congruence equation (1) is same as $f(p^2) - s_0 \equiv 0 \pmod{p^4}$. Therefore, it suffices to show that $s_1p^2 \equiv 0 \pmod{p^4}$ or $s_1 \equiv 0 \pmod{p^2}$.

Since $a^{p-1} \equiv 1 \pmod{p}$ for all $1 \leq a \leq p-1$, we can factor

$$x^{p-1} - 1 \equiv (x-1)(x-2)\cdots(x-(p-1)) \pmod{p}. \quad (3)$$

Comparing the coefficients of the left hand side of (3) with those of the right hand side of (2), we obtain $p \mid s_i$ for all $1 \leq i \leq p-2$ and $s_0 \equiv -1 \pmod{p}$. On the other hand, plugging p for x in (2), we get

$$f(p) = (p-1)! = s_0 = p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_1p + s_0,$$

which implies

$$p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_2p^2 = -s_1p.$$

Since $p \geq 5$, $p \mid s_2$ and hence $s_1 \equiv 0 \pmod{p^2}$ as desired.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB . Let O_1 be the circle tangent to the line AB at P and tangent to the circle O . Let ℓ be the tangent line, different from the line AB , to O_1 passing through A . Let C be the intersection point, different from A , of ℓ and O . Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC . Prove that the circle O_2 is tangent to the circle O .

(Solution) Let S be the tangent point of the circles O and O_1 and let T be the intersection point, different from S , of the circle O and the line SP . Let X be the tangent point of ℓ to O_1 and let M be the midpoint of the line segment XP . Since $\angle TBP = \angle ASP$, the triangle TBP is similar to the triangle ASP . Therefore,

$$\frac{PT}{PB} = \frac{PA}{PS}.$$

Since the line ℓ is tangent to the circle O_1 at X , we have

$$\angle SPX = 90^\circ - \angle XSP = 90^\circ - \angle APM = \angle PAM$$

which implies that the triangle PAM is similar to the triangle SPX . Consequently,

$$\frac{XS}{XP} = \frac{MP}{MA} = \frac{XP}{2MA} \quad \text{and} \quad \frac{XP}{PS} = \frac{MA}{AP}.$$

From this and the above observation follows

$$\frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{XP}{2MA} \cdot \frac{PA}{PS} = \frac{XP}{2MA} \cdot \frac{MA}{XP} = \frac{1}{2}. \quad (1)$$

Let A' be the intersection point of the circle O and the perpendicular bisector of the chord BC such that A, A' are on the same side of the line BC , and N be the intersection point of the lines $A'Q$ and CT . Since

$$\angle NCQ = \angle TCB = \angle TCA = \angle TBA = \angle TBP$$

and

$$\angle CA'Q = \frac{\angle CAB}{2} = \frac{\angle XAP}{2} = \angle PAM = \angle SPX,$$

the triangle NCQ is similar to the triangle TBP and the triangle $CA'Q$ is similar to the triangle SPX . Therefore

$$\frac{QN}{QC} = \frac{PT}{PB} \quad \text{and} \quad \frac{QC}{QA'} = \frac{XS}{XP}.$$

and hence $QA' = 2QN$ by (1). This implies that N is the midpoint of the line segment QA' . Let the circle O_2 touch the line segment AC at Y . Since

$$\angle ACN = \angle ACT = \angle BCT = \angle QCN$$

and $|CY| = |CQ|$, the triangles YCN and QCN are congruent and hence $NY \perp AC$ and $NY = NQ = NA'$. Therefore, N is the center of the circle O_2 , which completes the proof.

Remark: Analytic solutions are possible: For example, one can prove for a triangle ABC inscribed in a circle O that $AB = k(2 + 2t)$, $AC = k(1 + 2t)$, $BC = k(1 + 4t)$ for some positive numbers k, t if and only if there exists a circle O_1 such that O_1 is tangent to the side AB at its midpoint, the side AC and the circle O .

One obtains $AB = k'(1 + 4t')$, $AC = k'(1 + 2t')$, $BC = k'(2 + 2t')$ by substituting $t = 1/4t'$ and $k = 2k't'$. So, there exists a circle O_2 such that O_2 is tangent to the side BC at its midpoint, the side AC and the circle O .

In the above, $t = \tan^2 \alpha$ and $k = \frac{4R \tan \alpha}{(1 + \tan^2 \alpha)(1 + 4 \tan^2 \alpha)}$, where R is the radius of O and $\angle A = 2\alpha$. Furthermore, $t' = \tan^2 \gamma$ and $k' = \frac{4R \tan \gamma}{(1 + \tan^2 \gamma)(1 + 4 \tan^2 \gamma)}$, where $\angle C = 2\gamma$. Observe that $\sqrt{tt'} = \tan \alpha \cdot \tan \gamma = \frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{1}{2}$, which implies $tt' = \frac{1}{4}$. It is now routine easy to check that $k = 2k't'$.

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set

of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

(Solution) Let C be the set of n clowns. Label the colours $1, 2, 3, \dots, 12$. For each $i = 1, 2, \dots, 12$, let E_i denote the set of clowns who use colour i . For each subset S of $\{1, 2, \dots, 12\}$, let E_S be the set of clowns who use exactly those colours in S . Since $S \neq S'$ implies $E_S \cap E_{S'} = \emptyset$, we have

$$\sum_S |E_S| = |C| = n,$$

where S runs over all subsets of $\{1, 2, \dots, 12\}$. Now for each i ,

$$E_S \subseteq E_i \quad \text{if and only if} \quad i \in S,$$

and hence

$$|E_i| = \sum_{i \in S} |E_S|.$$

By assumption, we know that $|E_i| \leq 20$ and that if $E_S \neq \emptyset$, then $|S| \geq 5$. From this we obtain

$$20 \times 12 \geq \sum_{i=1}^{12} |E_i| = \sum_{i=1}^{12} \left(\sum_{i \in S} |E_S| \right) \geq 5 \sum_S |E_S| = 5n.$$

Therefore $n \leq 48$.

Now, define a sequence $\{c_i\}_{i=1}^{52}$ of colours in the following way:

$$\begin{array}{cccc|cccc|cccc|} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & | \\ 4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 & 12 & 9 & 10 & 11 & | \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & | \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 9 & | 1 & 2 & 3 & 4 \end{array}$$

The first row lists c_1, \dots, c_{12} in order, the second row lists c_{13}, \dots, c_{24} in order, the third row lists c_{25}, \dots, c_{36} in order, and finally the last row lists c_{37}, \dots, c_{52} in order. For each j , $1 \leq j \leq 48$, assign colours $c_j, c_{j+1}, c_{j+2}, c_{j+3}, c_{j+4}$ to the j -th clown. It is easy to check that this assignment satisfies all conditions given above. So, 48 is the largest for n .

Remark: The fact that $n \leq 48$ can be obtained in a much simpler observation that

$$5n \leq 12 \times 20 = 240.$$

There are many other ways of constructing 48 distinct sets consisting of 5 colours. For example, consider the sets

$$\begin{array}{cccc} \{1, 2, 3, 4, 5, 6\}, & \{3, 4, 5, 6, 7, 8\}, & \{5, 6, 7, 8, 9, 10\}, & \{7, 8, 9, 10, 11, 12\}, \\ \{9, 10, 11, 12, 1, 2\}, & \{11, 12, 1, 2, 3, 4\}, & \{1, 2, 5, 6, 9, 10\}, & \{3, 4, 7, 8, 11, 12\}. \end{array}$$

Each of the above 8 sets has 6 distinct subsets consisting of exactly 5 colours. It is easy to check that the 48 subsets obtained in this manner are all distinct.

XIX Asian Pacific Mathematics Olympiad



Time allowed: 4 hours

Each problem is worth 7 points

** The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC . Prove that

$$2\angle AHI = 3\angle ABC.$$

Problem 3. Consider n disks C_1, C_2, \dots, C_n in a plane such that for each $1 \leq i < n$, the center of C_i is on the circumference of C_{i+1} , and the center of C_n is on the circumference of C_1 . Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j . Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \geq 1.$$

Problem 5. A regular (5×5) -array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

XIX Asian Pacific Mathematics Olympiad



Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Solution. Without loss of generality, we may assume that S contains only positive integers. Let

$$S = \{2^{a_i}3^{b_i} \mid a_i, b_i \in \mathbb{Z}, a_i, b_i \geq 0, 1 \leq i \leq 9\}.$$

It suffices to show that there are $1 \leq i_1, i_2, i_3 \leq 9$ such that

$$a_{i_1} + a_{i_2} + a_{i_3} \equiv b_{i_1} + b_{i_2} + b_{i_3} \equiv 0 \pmod{3}. \quad (\dagger)$$

For $n = 2^a3^b \in S$, let's call $(a \pmod{3}, b \pmod{3})$ the *type* of n . Then there are 9 possible types:

$$(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2).$$

Let $N(i, j)$ be the number of integers in S of type (i, j) . We obtain 3 distinct integers whose product is a perfect cube when

- (1) $N(i, j) \geq 3$ for some i, j , or
- (2) $N(i, 0)N(i, 1)N(i, 2) \neq 0$ for some $i = 0, 1, 2$, or
- (3) $N(0, j)N(1, j)N(2, j) \neq 0$ for some $j = 0, 1, 2$, or
- (4) $N(i_1, j_1)N(i_2, j_2)N(i_3, j_3) \neq 0$, where $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{0, 1, 2\}$.

Assume that none of the conditions (1)~(3) holds. Since $N(i, j) \leq 2$ for all (i, j) , there are at least five $N(i, j)$'s that are nonzero. Furthermore, among those nonzero $N(i, j)$'s, no three have the same i nor the same j . Using these facts, one may easily conclude that the condition (4) should hold. (For example, if one places each nonzero $N(i, j)$ in the (i, j) -th box of a regular 3×3 array of boxes whose rows and columns are indexed by 0, 1 and 2, then one can always find three boxes, occupied by at least one nonzero $N(i, j)$, whose rows and columns are all distinct. This implies (4).)

Second solution. Up to (\dagger), we do the same as above and get 9 possible types :

$$(a \pmod 3, b \pmod 3) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)$$

for $n = 2^a 3^b \in S$.

Note that (i) among any 5 integers, there exist 3 whose sum is $0 \pmod 3$, and that (ii) if $i, j, k \in \{0, 1, 2\}$, then $i + j + k \equiv 0 \pmod 3$ if and only if $i = j = k$ or $\{i, j, k\} = \{0, 1, 2\}$.

Let's define

T : the set of types of the integers in S ;

$N(i)$: the number of integers in S of the type (i, \cdot) ;

$M(i)$: the number of integers $j \in \{0, 1, 2\}$ such that $(i, j) \in T$.

If $N(i) \geq 5$ for some i , the result follows from (i). Otherwise, for some permutation (i, j, k) of $(0, 1, 2)$,

$$N(i) \geq 3, \quad N(j) \geq 3, \quad N(k) \geq 1.$$

If $M(i)$ or $M(j)$ is 1 or 3, the result follows from (ii). Otherwise $M(i) = M(j) = 2$. Then either

$$(i, x), (i, y), (j, x), (j, y) \in T \quad \text{or} \quad (i, x), (i, y), (j, x), (j, z) \in T$$

for some permutation (x, y, z) of $(0, 1, 2)$. Since $N(k) \geq 1$, at least one of (k, x) , (k, y) and (k, z) contained in T . Therefore, in any case, the result follows from (ii). (For example, if $(k, y) \in T$, then take $(i, y), (j, y), (k, y)$ or $(i, x), (j, z), (k, y)$ from T .)

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC . Prove that

$$2\angle AHI = 3\angle ABC.$$

Solution. Let D be the intersection point of the lines AH and BC . Let K be the intersection point of the circumcircle O of the triangle ABC and the line AH . Let the line through I perpendicular to BC meet BC and the minor arc BC of the circumcircle O at E and N , respectively. We have

$$\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 180^\circ - \frac{1}{2}(\angle ABC + \angle ACB) = 90^\circ + \frac{1}{2}\angle BAC = 120^\circ$$

and also $\angle BNC = 180^\circ - \angle BAC = 120^\circ = \angle BIC$. Since $IN \perp BC$, the quadrilateral $BICN$ is a kite and thus $IE = EN$.

Now, since H is the orthocenter of the triangle ABC , $HD = DK$. Also because $ED \perp IN$ and $ED \perp HK$, we conclude that $IHK N$ is an isosceles trapezoid with $IH = NK$.

Hence

$$\angle AHI = 180^\circ - \angle IHK = 180^\circ - \angle AKN = \angle ABN.$$

Since $IE = EN$ and $BE \perp IN$, the triangles IBE and NBE are congruent. Therefore

$$\angle NBE = \angle IBE = \angle IBC = \angle IBA = \frac{1}{2}\angle ABC$$

and thus

$$\angle AHI = \angle ABN = \frac{3}{2}\angle ABC.$$

Second solution. Let P, Q and R be the intersection points of BH, CH and AH with AC, AB and BC , respectively. Then we have $\angle IBH = \angle ICH$. Indeed,

$$\angle IBH = \angle ABP - \angle ABI = 30^\circ - \frac{1}{2}\angle ABC$$

and

$$\angle ICH = \angle ACI - \angle ACH = \frac{1}{2}\angle ACB - 30^\circ = 30^\circ - \frac{1}{2}\angle ABC,$$

because $\angle ABH = \angle ACH = 30^\circ$ and $\angle ACB + \angle ABC = 120^\circ$. (Note that $\angle ABP > \angle ABI$ and $\angle ACI > \angle ACH$ because AB is the longest side of the triangle ABC under the given conditions.) Therefore $BIHC$ is a cyclic quadrilateral and thus

$$\angle BHI = \angle BCI = \frac{1}{2}\angle ACB.$$

On the other hand,

$$\angle BHR = 90^\circ - \angle HBR = 90^\circ - (\angle ABC - \angle ABH) = 120^\circ - \angle ABC.$$

Therefore,

$$\begin{aligned}\angle AHI &= 180^\circ - \angle BHI - \angle BHR = 60^\circ - \frac{1}{2}\angle ACB + \angle ABC \\ &= 60^\circ - \frac{1}{2}(120^\circ - \angle ABC) + \angle ABC = \frac{3}{2}\angle ABC.\end{aligned}$$

Problem 3. Consider n disks C_1, C_2, \dots, C_n in a plane such that for each $1 \leq i < n$, the center of C_i is on the circumference of C_{i+1} , and the center of C_n is on the circumference of C_1 . Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j . Determine the maximum possible score.

Solution. The answer is $(n-1)(n-2)/2$.

Let's call a set of n disks satisfying the given conditions an *n-configuration*. For an n -configuration $\mathcal{C} = \{C_1, \dots, C_n\}$, let $S_{\mathcal{C}} = \{(i, j) \mid C_i \text{ properly contains } C_j\}$. So, the score of an n -configuration \mathcal{C} is $|S_{\mathcal{C}}|$.

We'll show that (i) there is an n -configuration \mathcal{C} for which $|S_{\mathcal{C}}| = (n-1)(n-2)/2$, and that (ii) $|S_{\mathcal{C}}| \leq (n-1)(n-2)/2$ for any n -configuration \mathcal{C} .

Let C_1 be any disk. Then for $i = 2, \dots, n-1$, take C_i inside C_{i-1} so that the circumference of C_i contains the center of C_{i-1} . Finally, let C_n be a disk whose center is on the circumference of C_1 and whose circumference contains the center of C_{n-1} . This gives $S_{\mathcal{C}} = \{(i, j) \mid 1 \leq i < j \leq n-1\}$ of size $(n-1)(n-2)/2$, which proves (i).

For any n -configuration \mathcal{C} , $S_{\mathcal{C}}$ must satisfy the following properties:

- (1) $(i, i) \notin S_{\mathcal{C}}$,
- (2) $(i+1, i) \notin S_{\mathcal{C}}$, $(1, n) \notin S_{\mathcal{C}}$,
- (3) if $(i, j), (j, k) \in S_{\mathcal{C}}$, then $(i, k) \in S_{\mathcal{C}}$,
- (4) if $(i, j) \in S_{\mathcal{C}}$, then $(j, i) \notin S_{\mathcal{C}}$.

Now we show that a set G of ordered pairs of integers between 1 and n , satisfying the conditions (1)~(4), can have no more than $(n-1)(n-2)/2$ elements. Suppose that there exists a set G that satisfies the conditions (1)~(4), and has more than $(n-1)(n-2)/2$ elements. Let n be the least positive integer with which there exists such a set G . Note that G must have $(i, i+1)$ for some $1 \leq i \leq n$ or $(n, 1)$, since otherwise G can have at most

$$\binom{n}{2} - n = \frac{n(n-3)}{2} < \frac{(n-1)(n-2)}{2}$$

elements. Without loss of generality we may assume that $(n, 1) \in G$. Then $(1, n-1) \notin G$, since otherwise the condition (3) yields $(n, n-1) \in G$ contradicting the condition (2). Now let $G' = \{(i, j) \in G \mid 1 \leq i, j \leq n-1\}$, then G' satisfies the conditions (1)~(4), with $n-1$.

We now claim that $|G - G'| \leq n-2$:

Suppose that $|G - G'| > n-2$, then $|G - G'| = n-1$ and hence for each $1 \leq i \leq n-1$, either (i, n) or (n, i) must be in G . We already know that $(n, 1) \in G$ and $(n-1, n) \in G$ (because $(n, n-1) \notin G$) and this implies that $(n, n-2) \notin G$ and $(n-2, n) \in G$. If we keep doing this process, we obtain $(1, n) \in G$, which is a contradiction.

Since $|G - G'| \leq n - 2$, we obtain

$$|G'| \geq \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-2)(n-3)}{2}.$$

This, however, contradicts the minimality of n , and hence proves (ii).

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \geq 1.$$

Solution. We first note that

$$\begin{aligned} \frac{x^2 + yz}{\sqrt{2x^2(y+z)}} &= \frac{x^2 - x(y+z) + yz}{\sqrt{2x^2(y+z)}} + \frac{x(y+z)}{\sqrt{2x^2(y+z)}} \\ &= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} + \sqrt{\frac{y+z}{2}} \\ &\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} + \frac{\sqrt{y} + \sqrt{z}}{2}. \end{aligned} \quad (1)$$

Similarly, we have

$$\frac{y^2 + zx}{\sqrt{2y^2(z+x)}} \geq \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}} + \frac{\sqrt{z} + \sqrt{x}}{2}, \quad (2)$$

$$\frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \geq \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} + \frac{\sqrt{x} + \sqrt{y}}{2}. \quad (3)$$

We now add (1)~(3) to get

$$\begin{aligned} &\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \\ &\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} + \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}} + \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} + \sqrt{x} + \sqrt{y} + \sqrt{z} \\ &= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} + \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}} + \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} + 1. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} + \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}} + \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} \geq 0. \quad (4)$$

Now, assume without loss of generality, that $x \geq y \geq z$. Then we have

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} \geq 0$$

and

$$\begin{aligned} & \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} + \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}} = \frac{(y-z)(x-z)}{\sqrt{2z^2(x+y)}} - \frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}} \\ & \geq \frac{(y-z)(x-y)}{\sqrt{2z^2(x+y)}} - \frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}} = (y-z)(x-y) \left(\frac{1}{\sqrt{2z^2(x+y)}} - \frac{1}{\sqrt{2y^2(z+x)}} \right). \end{aligned}$$

The last quantity is non-negative due to the fact that

$$y^2(z+x) = y^2z + y^2x \geq yz^2 + z^2x = z^2(x+y).$$

This completes the proof.

Second solution. By Cauchy-Schwarz inequality,

$$\begin{aligned} & \left(\frac{x^2}{\sqrt{2x^2(y+z)}} + \frac{y^2}{\sqrt{2y^2(z+x)}} + \frac{z^2}{\sqrt{2z^2(x+y)}} \right) \\ & \times (\sqrt{2(y+z)} + \sqrt{2(z+x)} + \sqrt{2(x+y)}) \geq (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 = 1, \end{aligned} \quad (5)$$

and

$$\begin{aligned} & \left(\frac{yz}{\sqrt{2x^2(y+z)}} + \frac{zx}{\sqrt{2y^2(z+x)}} + \frac{xy}{\sqrt{2z^2(x+y)}} \right) \\ & \times (\sqrt{2(y+z)} + \sqrt{2(z+x)} + \sqrt{2(x+y)}) \geq \left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \right)^2. \end{aligned} \quad (6)$$

We now combine (5) and (6) to find

$$\begin{aligned} & \left(\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \right) \\ & \times (\sqrt{2(x+y)} + \sqrt{2(y+z)} + \sqrt{2(z+x)}) \\ & \geq 1 + \left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \right)^2 \geq 2 \left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \right). \end{aligned}$$

Thus, it suffices to show that

$$2 \left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \right) \geq \sqrt{2(y+z)} + \sqrt{2(z+x)} + \sqrt{2(x+y)}. \quad (7)$$

Consider the following inequality using AM-GM inequality

$$\left[\sqrt{\frac{yz}{x}} + \left(\frac{1}{2} \sqrt{\frac{zx}{y}} + \frac{1}{2} \sqrt{\frac{xy}{z}} \right) \right]^2 \geq 4 \sqrt{\frac{yz}{x}} \left(\frac{1}{2} \sqrt{\frac{zx}{y}} + \frac{1}{2} \sqrt{\frac{xy}{z}} \right) = 2(y+z),$$

or equivalently

$$\sqrt{\frac{yz}{x}} + \left(\frac{1}{2}\sqrt{\frac{zx}{y}} + \frac{1}{2}\sqrt{\frac{xy}{z}} \right) \geq \sqrt{2(y+z)}.$$

Similarly, we have

$$\begin{aligned} \sqrt{\frac{zx}{y}} + \left(\frac{1}{2}\sqrt{\frac{xy}{z}} + \frac{1}{2}\sqrt{\frac{yz}{x}} \right) &\geq \sqrt{2(z+x)}, \\ \sqrt{\frac{xy}{z}} + \left(\frac{1}{2}\sqrt{\frac{yz}{x}} + \frac{1}{2}\sqrt{\frac{zx}{y}} \right) &\geq \sqrt{2(x+y)}. \end{aligned}$$

Adding the last three inequalities, we get

$$2 \left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}} \right) \geq \sqrt{2(y+z)} + \sqrt{2(z+x)} + \sqrt{2(x+y)}.$$

This completes the proof.

Problem 5. A regular (5×5) -array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

Solution. We assign the following *first labels* to the 25 positions of the lights:

1	1	0	1	1
0	0	0	0	0
1	1	0	1	1
0	0	0	0	0
1	1	0	1	1

For each on-off combination of lights in the array, define its *first value* to be the sum of the first labels of those positions at which the lights are switched on. It is easy to check that toggling any switch always leads to an on-off combination of lights whose first value has the same parity (the remainder when divided by 2) as that of the previous on-off combination.

The 90° rotation of the first labels gives us another labels (let us call it the *second labels*) which also makes the parity of the *second value* (the sum of the second labels of those positions at which the lights are switched on) invariant under toggling.

1	0	1	0	1
1	0	1	0	1
0	0	0	0	0
1	0	1	0	1
1	0	1	0	1

Since the parity of the first and the second values of the initial status is 0, after certain number of toggles the parity must remain unchanged with respect to the first labels and the second labels as well. Therefore, if exactly one light is on after some number of toggles, the label of that position must be 0 with respect to both labels. Hence according to the above pictures, the possible positions are the ones marked with $*$'s in the following picture:

	*2		*1	
		*0		
	*3		*4	

Now we demonstrate that all five positions are possible :

Toggling the positions checked by t (the order of toggling is irrelevant) in the first picture makes the center($*_0$) the only position with light on and the second picture makes the position $*_1$ the only position with light on. The other $*_i$'s can be obtained by rotating the second picture appropriately.

			t	t
		t		
	t	t		t
t				t
t		t	t	

	t		t	
t	t		t	t
	t			
		t	t	t
			t	

XX Asian Pacific Mathematics Olympiad



March, 2008

Time allowed: 4 hours

Each problem is worth 7 points

** The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC , respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC , respectively. Prove that $\angle BPC < 120^\circ$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let Γ be the circumcircle of a triangle ABC . A circle passing through points A and C meets the sides BC and BA at D and E , respectively. The lines AD and CE meet Γ again at G and H , respectively. The tangent lines of Γ at A and C meet the line DE at L and M , respectively. Prove that the lines LH and MG meet at Γ .

Problem 4. Consider the function $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, where \mathbb{N}_0 is the set of all non-negative integers, defined by the following conditions:

(i) $f(0) = 0$, (ii) $f(2n) = 2f(n)$ and (iii) $f(2n + 1) = n + 2f(n)$ for all $n \geq 0$.

(a) Determine the three sets $L := \{n \mid f(n) < f(n + 1)\}$, $E := \{n \mid f(n) = f(n + 1)\}$, and $G := \{n \mid f(n) > f(n + 1)\}$.

(b) For each $k \geq 0$, find a formula for $a_k := \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k .

Problem 5. Let a, b, c be integers satisfying $0 < a < c - 1$ and $1 < b < c$. For each k , $0 \leq k \leq a$, let r_k , $0 \leq r_k < c$, be the remainder of kb when divided by c . Prove that the two sets $\{r_0, r_1, r_2, \dots, r_a\}$ and $\{0, 1, 2, \dots, a\}$ are different.

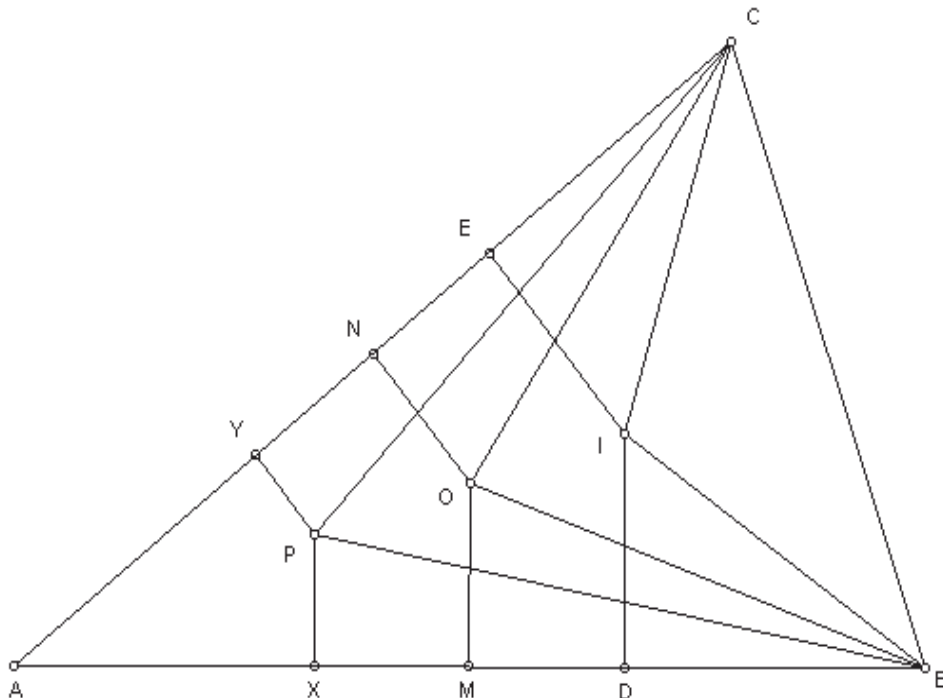
XX Asian Pacific Mathematics Olympiad



March, 2008

Problem 1. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC , respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC , respectively. Prove that $\angle BPC < 120^\circ$.

(Solution) Let I be the incenter of $\triangle ABC$, and let the feet of the perpendiculars from I to AB and to AC be D and E , respectively. (Without loss of generality, we may assume that AC is the longest side. Then X lies on the line segment AD . Although P may or may not lie inside $\triangle ABC$, the proof below works for both cases. Note that P is on the line perpendicular to AB passing through X .) Let O be the midpoint of IP , and let the feet of the perpendiculars from O to AB and to AC be M and N , respectively. Then M and N are the midpoints of DX and EY , respectively.



The conditions on the points X and Y yield the equations

$$AX = \frac{AB + BC - CA}{2} \quad \text{and} \quad AY = \frac{BC + CA - AB}{2}.$$

From $AD = AE = \frac{CA + AB - BC}{2}$, we obtain

$$BD = AB - AD = AB - \frac{CA + AB - BC}{2} = \frac{AB + BC - CA}{2} = AX.$$

Since M is the midpoint of DX , it follows that M is the midpoint of AB . Similarly, N is the midpoint of AC . Therefore, the perpendicular bisectors of AB and AC meet at O , that is, O is the circumcenter of $\triangle ABC$. Since $\angle BAC < 60^\circ$, O lies on the same side of BC as the point A and

$$\angle BOC = 2\angle BAC.$$

We can compute $\angle BIC$ as follows:

$$\begin{aligned} \angle BIC &= 180^\circ - \angle IBC - \angle ICB = 180^\circ - \frac{1}{2}\angle ABC - \frac{1}{2}\angle ACB \\ &= 180^\circ - \frac{1}{2}(\angle ABC + \angle ACB) = 180^\circ - \frac{1}{2}(180^\circ - \angle BAC) = 90^\circ + \frac{1}{2}\angle BAC \end{aligned}$$

It follows from $\angle BAC < 60^\circ$ that

$$2\angle BAC < 90^\circ + \frac{1}{2}\angle BAC, \quad \text{i.e.,} \quad \angle BOC < \angle BIC.$$

From this it follows that I lies inside the circumcircle of the isosceles triangle BOC because O and I lie on the same side of BC . However, as O is the midpoint of IP , P must lie outside the circumcircle of triangle BOC and on the same side of BC as O . Therefore

$$\angle BPC < \angle BOC = 2\angle BAC < 120^\circ.$$

Remark. If one assumes that $\angle A$ is smaller than the other two, then it is clear that the line PX (or the line perpendicular to AB at X if $P = X$) runs through the excenter I_C of the excircle tangent to the side AB . Since $2\angle ACI_C = \angle ACB$ and $BC < AC$, we have $2\angle PCB > \angle C$. Similarly, $2\angle PBC > \angle B$. Therefore,

$$\angle BPC = 180^\circ - (\angle PBC + \angle PCB) < 180^\circ - \left(\frac{\angle B + \angle C}{2}\right) = 90^\circ + \frac{\angle A}{2} < 120^\circ.$$

In this way, a special case of the problem can be easily proved.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

(Solution) We let C be the set of all 46 students in the class and let

$$s := \max\{|S| : S \subseteq C \text{ such that } S \text{ contains no group properly}\}.$$

Then it suffices to prove that $s \geq 10$. (If $|S| = s > 10$, we may choose a subset of S consisting of 10 students.)

Suppose that $s \leq 9$ and let S be a set of size s in which no group is properly contained. Take any student, say v , from outside S . Because of the maximality of s , there should be a group containing the student v and two other students in S . The number of ways to choose two students from S is

$$\binom{s}{2} \leq \binom{9}{2} = 36.$$

On the other hand, there are at least $37 = 46 - 9$ students outside of S . Thus, among those 37 students outside, there is at least one student, say u , who does not belong to any group containing two students in S and one outside. This is because no two distinct groups have two members in common. But then, S can be enlarged by including u , which is a contradiction.

Remark. One may choose a subset S of C that contains no group properly. Then, assuming $|S| < 10$, prove that there is a student outside S , say u , who does not belong to any group containing two students in S . After enlarging S by including u , prove that the enlarged S still contains no group properly.

Problem 3. Let Γ be the circumcircle of a triangle ABC . A circle passing through points A and C meets the sides BC and BA at D and E , respectively. The lines AD and CE meet Γ again at G and H , respectively. The tangent lines of Γ at A and C meet the line DE at L and M , respectively. Prove that the lines LH and MG meet at Γ .

(Solution) Let MG meet Γ at P . Since $\angle MCD = \angle CAE$ and $\angle MDC = \angle CAE$, we have $MC = MD$. Thus

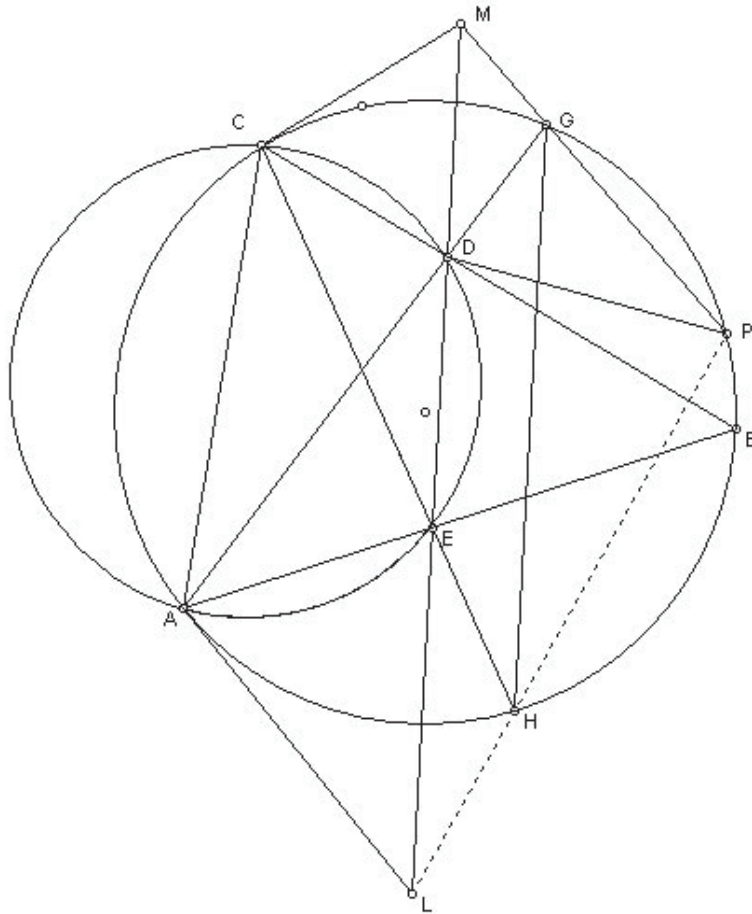
$$MD^2 = MC^2 = MG \cdot MP$$

and hence MD is tangent to the circumcircle of $\triangle DGP$. Therefore $\angle DGP = \angle EDP$.

Let Γ' be the circumcircle of $\triangle BDE$. If $B = P$, then, since $\angle BGD = \angle BDE$, the tangent lines of Γ' and Γ at B should coincide, that is Γ' is tangent to Γ from inside. Let $B \neq P$. If P lies in the same side of the line BC as G , then we have

$$\angle EDP + \angle ABP = 180^\circ$$

because $\angle DGP + \angle ABP = 180^\circ$. That is, the quadrilateral $BPDE$ is cyclic, and hence P is on the intersection of Γ' with Γ .



Otherwise,

$$\angle EDP = \angle DGP = \angle AGP = \angle ABP = \angle EBP.$$

Therefore the quadrilateral $PBDE$ is cyclic, and hence P again is on the intersection of Γ' with Γ .

Similarly, if LH meets Γ at Q , we either have $Q = B$, in which case Γ' is tangent to Γ from inside, or $Q \neq B$. In the latter case, Q is on the intersection of Γ' with Γ . In either case, we have $P = Q$.

Problem 4. Consider the function $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, where \mathbb{N}_0 is the set of all non-negative integers, defined by the following conditions:

$$(i) f(0) = 0, \quad (ii) f(2n) = 2f(n) \quad \text{and} \quad (iii) f(2n+1) = n + 2f(n) \quad \text{for all } n \geq 0.$$

(a) Determine the three sets $L := \{n \mid f(n) < f(n+1)\}$, $E := \{n \mid f(n) = f(n+1)\}$, and $G := \{n \mid f(n) > f(n+1)\}$.

(b) For each $k \geq 0$, find a formula for $a_k := \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k .

(Solution) (a) Let

$$L_1 := \{2k : k > 0\}, \quad E_1 := \{0\} \cup \{4k+1 : k \geq 0\}, \quad \text{and} \quad G_1 := \{4k+3 : k \geq 0\}.$$

We will show that $L_1 = L$, $E_1 = E$, and $G_1 = G$. It suffices to verify that $L_1 \subseteq L$, $E_1 \subseteq E$, and $G_1 \subseteq G$ because L_1 , E_1 , and G_1 are mutually disjoint and $L_1 \cup E_1 \cup G_1 = \mathbb{N}_0$.

Firstly, if $k > 0$, then $f(2k) - f(2k+1) = -k < 0$ and therefore $L_1 \subseteq L$.

Secondly, $f(0) = 0$ and

$$\begin{aligned} f(4k+1) &= 2k + 2f(2k) = 2k + 4f(k) \\ f(4k+2) &= 2f(2k+1) = 2(k + 2f(k)) = 2k + 4f(k) \end{aligned}$$

for all $k \geq 0$. Thus, $E_1 \subseteq E$.

Lastly, in order to prove $G_1 \subseteq G$, we claim that $f(n+1) - f(n) \leq n$ for all n . (In fact, one can prove a stronger inequality: $f(n+1) - f(n) \leq n/2$.) This is clearly true for even n from the definition since for $n = 2t$,

$$f(2t+1) - f(2t) = t \leq n.$$

If $n = 2t+1$ is odd, then (assuming inductively that the result holds for all nonnegative $m < n$), we have

$$\begin{aligned} f(n+1) - f(n) &= f(2t+2) - f(2t+1) = 2f(t+1) - t - 2f(t) \\ &= 2(f(t+1) - f(t)) - t \leq 2t - t = t < n. \end{aligned}$$

For all $k \geq 0$,

$$\begin{aligned} f(4k+4) - f(4k+3) &= f(2(2k+2)) - f(2(2k+1)+1) \\ &= 4f(k+1) - (2k+1 + 2f(2k+1)) = 4f(k+1) - (2k+1 + 2k + 4f(k)) \\ &= 4(f(k+1) - f(k)) - (4k+1) \leq 4k - (4k+1) < 0. \end{aligned}$$

This proves $G_1 \subseteq G$.

(b) Note that $a_0 = a_1 = f(1) = 0$. Let $k \geq 2$ and let $N_k = \{0, 1, 2, \dots, 2^k\}$. First we claim that the maximum a_k occurs at the largest number in $G \cap N_k$, that is, $a_k = f(2^k - 1)$. We use mathematical induction on k to prove the claim. Note that $a_2 = f(3) = f(2^2 - 1)$.

Now let $k \geq 3$. For every even number $2t$ with $2^{k-1} + 1 < 2t \leq 2^k$,

$$f(2t) = 2f(t) \leq 2a_{k-1} = 2f(2^{k-1} - 1) \tag{†}$$

by induction hypothesis. For every odd number $2t+1$ with $2^{k-1} + 1 \leq 2t+1 < 2^k$,

$$\begin{aligned} f(2t+1) &= t + 2f(t) \leq 2^{k-1} - 1 + 2f(t) \\ &\leq 2^{k-1} - 1 + 2a_{k-1} = 2^{k-1} - 1 + 2f(2^{k-1} - 1) \end{aligned} \tag{‡}$$

again by induction hypothesis. Combining (†), (‡) and

$$f(2^k - 1) = f(2(2^{k-1} - 1) + 1) = 2^{k-1} - 1 + 2f(2^{k-1} - 1),$$

we may conclude that $a_k = f(2^k - 1)$ as desired.

Furthermore, we obtain

$$a_k = 2a_{k-1} + 2^{k-1} - 1$$

for all $k \geq 3$. Note that this recursive formula for a_k also holds for $k \geq 0, 1$ and 2 . Unwinding this recursive formula, we finally get

$$\begin{aligned} a_k &= 2a_{k-1} + 2^{k-1} - 1 = 2(2a_{k-2} + 2^{k-2} - 1) + 2^{k-1} - 1 \\ &= 2^2 a_{k-2} + 2 \cdot 2^{k-1} - 2 - 1 = 2^2(2a_{k-3} + 2^{k-3} - 1) + 2 \cdot 2^{k-1} - 2 - 1 \\ &= 2^3 a_{k-3} + 3 \cdot 2^{k-1} - 2^2 - 2 - 1 \\ &\vdots \\ &= 2^k a_0 + k2^{k-1} - 2^{k-1} - 2^{k-2} - \dots - 2 - 1 \\ &= k2^{k-1} - 2^k + 1 \quad \text{for all } k \geq 0. \end{aligned}$$

Problem 5. Let a, b, c be integers satisfying $0 < a < c - 1$ and $1 < b < c$. For each k , $0 \leq k \leq a$, let r_k , $0 \leq r_k < c$, be the remainder of kb when divided by c . Prove that the two sets $\{r_0, r_1, r_2, \dots, r_a\}$ and $\{0, 1, 2, \dots, a\}$ are different.

(Solution) Suppose that two sets are equal. Then $\gcd(b, c) = 1$ and the polynomial

$$f(x) := (1 + x^b + x^{2b} + \dots + x^{ab}) - (1 + x + x^2 + \dots + x^{a-1} + x^a)$$

is divisible by $x^c - 1$. (This is because: $m = n + cq \implies x^m - x^n = x^{n+cq} - x^n = x^n(x^{cq} - 1)$ and $(x^{cq} - 1) = (x^c - 1)((x^c)^{q-1} + (x^c)^{q-2} + \dots + 1)$.) From

$$f(x) = \frac{x^{(a+1)b} - 1}{x^b - 1} - \frac{x^{a+1} - 1}{x - 1} = \frac{F(x)}{(x-1)(x^b-1)},$$

where $F(x) = x^{ab+b+1} + x^b + x^{a+1} - x^{ab+b} - x^{a+b+1} - x$, we have

$$F(x) \equiv 0 \pmod{x^c - 1}.$$

Since $x^c \equiv 1 \pmod{x^c - 1}$, we may conclude that

$$\{ab + b + 1, b, a + 1\} \equiv \{ab + b, a + b + 1, 1\} \pmod{c}. \quad (\dagger)$$

Thus,

$$b \equiv ab + b, a + b + 1 \text{ or } 1 \pmod{c}.$$

But neither $b \equiv 1 \pmod{c}$ nor $b \equiv a + b + 1 \pmod{c}$ are possible by the given conditions. Therefore, $b \equiv ab + b \pmod{c}$. But this is also impossible because $\gcd(b, c) = 1$.

XXI Asian Pacific Mathematics Olympiad



March, 2009

Time allowed: 4 hours

Each problem is worth 7 points

* *The contest problems are to be kept confidential until they are posted on the official APMO website (<http://www.kms.or.kr/Competitions/APMO>). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Consider the following operation on positive real numbers written on a blackboard: Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr .

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2 + 1} + \frac{a_2}{k^2 + 2} + \frac{a_3}{k^2 + 3} + \frac{a_4}{k^2 + 4} + \frac{a_5}{k^2 + 5} = \frac{1}{k^2} \quad \text{for } k = 1, 2, 3, 4, 5.$$

Find the value of $\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}$. (Express the value in a single fraction.)

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i . Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

Problem 4. Prove that for any positive integer k , there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \dots, \quad \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \dots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are all distinct.

Problem 5. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every ℓ kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

XXI Asian Pacific Mathematics Olympiad



March, 2009

Problem 1. Consider the following operation on positive real numbers written on a blackboard:

Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr .

(Solution) Using AM-GM inequality, we obtain

$$\frac{1}{r^2} = \frac{2}{ab} = \frac{2ab}{a^2b^2} \leq \frac{a^2 + b^2}{a^2b^2} \leq \frac{1}{a^2} + \frac{1}{b^2}. \quad (*)$$

Consequently, if we let S_ℓ be the sum of the squares of the reciprocals of the numbers written on the board after ℓ operations, then S_ℓ increases as ℓ increases, that is,

$$S_0 \leq S_1 \leq \cdots \leq S_{k^2-1}. \quad (**)$$

Therefore if we let s be the smallest real number written on the board after $k^2 - 1$ operations, then $\frac{1}{s^2} \geq \frac{1}{t^2}$ for any number t among k^2 numbers on the board and hence

$$k^2 \times \frac{1}{s^2} \geq S_{k^2-1} \geq S_0 = \frac{1}{r^2},$$

which implies that $s \leq kr$ as desired. □

Remark. The nature of the problem does not change at all if the numbers on the board are restricted to be positive integers. But that may mislead some contestants to think the problem is a number theoretic problem rather than a combinatorial problem.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2 + 1} + \frac{a_2}{k^2 + 2} + \frac{a_3}{k^2 + 3} + \frac{a_4}{k^2 + 4} + \frac{a_5}{k^2 + 5} = \frac{1}{k^2} \quad \text{for } k = 1, 2, 3, 4, 5.$$

Find the value of $\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}$. (Express the value in a single fraction.)

(Solution) Let $R(x) := \frac{a_1}{x^2 + 1} + \frac{a_2}{x^2 + 2} + \frac{a_3}{x^2 + 3} + \frac{a_4}{x^2 + 4} + \frac{a_5}{x^2 + 5}$. Then $R(\pm 1) = 1$, $R(\pm 2) = \frac{1}{4}$, $R(\pm 3) = \frac{1}{9}$, $R(\pm 4) = \frac{1}{16}$, $R(\pm 5) = \frac{1}{25}$ and $R(6)$ is the value to be found. Let's put $P(x) := (x^2 + 1)(x^2 + 2)(x^2 + 3)(x^2 + 4)(x^2 + 5)$ and $Q(x) := R(x)P(x)$. Then for $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get $Q(k) = R(k)P(k) = \frac{P(k)}{k^2}$, that is, $P(k) - k^2Q(k) = 0$. Since $P(x) - x^2Q(x)$ is a polynomial of degree 10 with roots $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get

$$P(x) - x^2Q(x) = A(x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16)(x^2 - 25). \quad (*)$$

Putting $x = 0$, we get $A = \frac{P(0)}{(-1)(-4)(-9)(-16)(-25)} = -\frac{1}{120}$. Finally, dividing both sides of (*) by $P(x)$ yields

$$1 - x^2R(x) = 1 - x^2\frac{Q(x)}{P(x)} = -\frac{1}{120} \cdot \frac{(x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16)(x^2 - 25)}{(x^2 + 1)(x^2 + 2)(x^2 + 3)(x^2 + 4)(x^2 + 5)}$$

and hence that

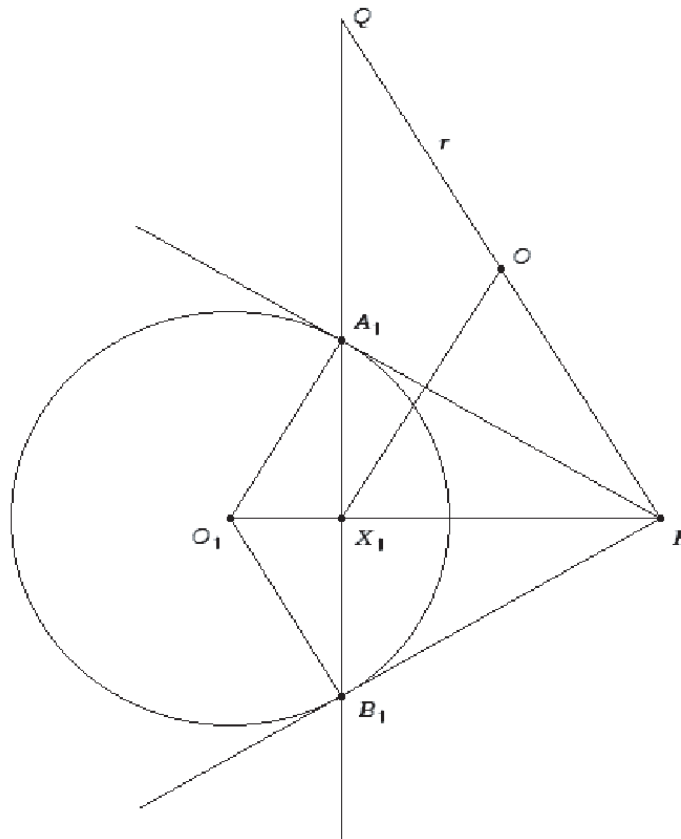
$$1 - 36R(6) = -\frac{35 \times 32 \times 27 \times 20 \times 11}{120 \times 37 \times 38 \times 39 \times 40 \times 41} = -\frac{3 \times 7 \times 11}{13 \times 19 \times 37 \times 41} = -\frac{231}{374699},$$

which implies $R(6) = \frac{187465}{6744582}$.

Remark. We can get $a_1 = \frac{1105}{72}$, $a_2 = -\frac{2673}{40}$, $a_3 = \frac{1862}{15}$, $a_4 = -\frac{1885}{18}$, $a_5 = \frac{1323}{40}$ by solving the given system of linear equations, which is extremely messy and takes a lot of time.

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i . Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

(Solution) Let O_i be the center and r_i the radius of circle Γ_i for each $i = 1, 2, 3$. Let P be an exceptional point, and let the three corresponding lines A_1B_1, A_2B_2, A_3B_3 concur at Q . Construct the circle with diameter PQ . Call the circle Γ , its center O and its radius r . We now claim that all exceptional points lie on Γ .



Let PO_1 intersect A_1B_1 in X_1 . As $PO_1 \perp A_1B_1$, we see that X_1 lies on Γ . As PA_1 is a tangent to Γ_1 , triangle PA_1O_1 is right-angled and similar to triangle $A_1X_1O_1$. It follows that

$$\frac{O_1X_1}{O_1A_1} = \frac{O_1A_1}{O_1P}, \quad \text{i.e.,} \quad O_1X_1 \cdot O_1P = O_1A_1^2 = r_1^2.$$

On the other hand, $O_1X_1 \cdot O_1P$ is also the power of O_1 with respect to Γ , so that

$$r_1^2 = O_1X_1 \cdot O_1P = (O_1O - r)(O_1O + r) = O_1O^2 - r^2, \quad (*)$$

and hence

$$r^2 = OO_1^2 - r_1^2 = (OO_1 - r_1)(OO_1 + r_1).$$

Thus, r^2 is the power of O with respect to Γ_1 . By the same token, r^2 is also the power of O with respect to Γ_2 and Γ_3 . Hence O must be the radical center of the three given circles. Since r , as the square root of the power of O with respect to the three given circles, does not depend on P , it follows that all exceptional points lie on Γ . \square

Remark. In the event of the radical point being at infinity (and hence the three radical axes being parallel), there are no exceptional points in the plane, which is consistent with the statement of the problem.

Problem 4. Prove that for any positive integer k , there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \dots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ are all distinct.

(Solution) For $k = 1$, there is nothing to prove. Henceforth assume $k \geq 2$.

Let p_1, p_2, \dots, p_k be k distinct primes such that

$$k < p_k < \dots < p_2 < p_1$$

and let $N = p_1 p_2 \dots p_k$. By Chinese Remainder Theorem, there exists a positive integer x satisfying

$$x \equiv -i \pmod{p_i}$$

for all $i = 1, 2, \dots, k$ and $x > N^2$. Consider the following sequence:

$$\frac{x+1}{N}, \frac{x+2}{N}, \dots, \frac{x+k}{N}.$$

This sequence is obviously an arithmetic sequence of positive rational numbers of length k . For each $i = 1, 2, \dots, k$, the numerator $x+i$ is divisible by p_i but not by p_j for $j \neq i$, for otherwise p_j divides $|i-j|$, which is not possible because $p_j > k > |i-j|$. Let

$$a_i := \frac{x+i}{p_i}, \quad b_i := \frac{N}{p_i} \quad \text{for all } i = 1, 2, \dots, k.$$

Then

$$\frac{x+i}{N} = \frac{a_i}{b_i}, \quad \gcd(a_i, b_i) = 1 \quad \text{for all } i = 1, 2, \dots, k,$$

and all b_i 's are distinct from each other. Moreover, $x > N^2$ implies

$$a_i = \frac{x+i}{p_i} > \frac{N^2}{p_i} > N > \frac{N}{p_j} = b_j \quad \text{for all } i, j = 1, 2, \dots, k$$

and hence all a_i 's are distinct from b_i 's. It only remains to show that all a_i 's are distinct from each other. This follows from

$$a_j = \frac{x+j}{p_j} > \frac{x+i}{p_j} > \frac{x+i}{p_i} = a_i \quad \text{for all } i < j$$

by our choice of p_1, p_2, \dots, p_k . Thus, the arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k}$$

of positive rational numbers satisfies the conditions of the problem. \square

Remark. Here is a much easier solution :

For any positive integer $k \geq 2$, consider the sequence

$$\frac{(k!)^2 + 1}{k!}, \frac{(k!)^2 + 2}{k!}, \dots, \frac{(k!)^2 + k}{k!}.$$

Note that $\gcd(k!, (k!)^2 + i) = i$ for all $i = 1, 2, \dots, k$. So, taking

$$a_i := \frac{(k!)^2 + i}{i}, \quad b_i := \frac{k!}{i} \quad \text{for all } i = 1, 2, \dots, k,$$

we have $\gcd(a_i, b_i) = 1$ and

$$a_i = \frac{(k!)^2 + i}{i} > a_j = \frac{(k!)^2 + j}{j} > b_i = \frac{k!}{i} > b_j = \frac{k!}{j}$$

for any $1 \leq i < j \leq k$. Therefore this sequence satisfies every condition given in the problem.

Problem 5. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every ℓ kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

(Solution) Let Zillis be d kilometers away from Argovia, where d is a positive real number. For simplicity, we will position Argovia at $(0, 0)$ and Zillis at $(d, 0)$, so that the car starts out facing east. We will investigate how the car moves around in the period of travelling the first ℓr kilometers, the second ℓr kilometers, \dots , and so on. We call each period of travelling ℓr kilometers a *section*. It is clear that the car will have identical behavior in every section except the direction of the car at the beginning.

Case 1: $\ell - r \equiv 2 \pmod{4}$. After the first section, the car has made $\ell - 1$ right turns and $r - 1$ left turns, which is a net of $2(\equiv \ell - r \pmod{4})$ right turns. Let the displacement vector for the first section be (x, y) . Since the car has rotated 180° , the displacement vector for the second section will be $(-x, -y)$, which will take the car back to $(0, 0)$ facing east again. We now have our original situation, and the car has certainly never travelled further than ℓr kilometers from Argovia. So, the car cannot reach Zillis if it is further apart from Argovia.

Case 2: $\ell - r \equiv 1 \pmod{4}$. After the first section, the car has made a net of 1 right turn. Let the displacement vector for the first section again be (x, y) . This time the car has rotated 90° clockwise. We can see that the displacements for the second, third and fourth section will be $(y, -x)$, $(-x, -y)$ and $(-y, x)$, respectively, so after four sections the car is back at $(0, 0)$ facing east. Since the car has certainly never travelled further than $2\ell r$ kilometers from Argovia, the car cannot reach Zillis if it is further apart from Argovia.

Case 3: $\ell - r \equiv 3 \pmod{4}$. An argument similar to that in **Case 2** (switching the roles of left and right) shows that the car cannot reach Zillis if it is further apart from Argovia.

Case 4: $\ell \equiv r \pmod{4}$. The car makes a net turn of 0° after each section, so it must be facing east. We are going to show that, after traversing the first section, the car will be at $(1, 0)$. It will be useful to interpret the Cartesian plane as the complex plane, i.e. writing $x + iy$ for (x, y) , where $i = \sqrt{-1}$. We will denote the k -th kilometer of movement by m_{k-1} ,

which takes values from the set $\{1, i, -1, -i\}$, depending on the direction. We then just have to show that

$$\sum_{k=0}^{\ell r-1} m_k = 1,$$

which implies that the car will get to Zillis no matter how far it is apart from Argovia.

Case 4a: $\ell \equiv r \equiv 1 \pmod{4}$. First note that for $k = 0, 1, \dots, \ell r - 1$,

$$m_k = i^{\lfloor k/\ell \rfloor} (-i)^{\lfloor k/r \rfloor}$$

since $\lfloor k/\ell \rfloor$ and $\lfloor k/r \rfloor$ are the exact numbers of left and right turns before the $(k+1)$ st kilometer, respectively. Let $a_k (\equiv k \pmod{\ell})$ and $b_k (\equiv k \pmod{r})$ be the remainders of k when divided by ℓ and r , respectively. Then, since

$$a_k = k - \left\lfloor \frac{k}{\ell} \right\rfloor \ell \equiv k - \left\lfloor \frac{k}{\ell} \right\rfloor \pmod{4} \quad \text{and} \quad b_k = k - \left\lfloor \frac{k}{r} \right\rfloor r \equiv k - \left\lfloor \frac{k}{r} \right\rfloor \pmod{4},$$

we have $\lfloor k/\ell \rfloor \equiv k - a_k \pmod{4}$ and $\lfloor k/r \rfloor \equiv k - b_k \pmod{4}$. We therefore have

$$m_k = i^{k-a_k} (-i)^{k-b_k} = (-i^2)^k i^{-a_k} (-i)^{-b_k} = (-i)^{a_k} i^{b_k}.$$

As ℓ and r are relatively prime, by Chinese Remainder Theorem, there is a bijection between pairs $(a_k, b_k) = (k \pmod{\ell}, k \pmod{r})$ and the numbers $k = 0, 1, 2, \dots, \ell r - 1$. Hence

$$\sum_{k=0}^{\ell r-1} m_k = \sum_{k=0}^{\ell r-1} (-i)^{a_k} i^{b_k} = \left(\sum_{k=0}^{\ell-1} (-i)^{a_k} \right) \left(\sum_{k=0}^{r-1} i^{b_k} \right) = 1 \times 1 = 1$$

as required because $\ell \equiv r \equiv 1 \pmod{4}$.

Case 4b: $\ell \equiv r \equiv 3 \pmod{4}$. In this case, we get

$$m_k = i^{a_k} (-i)^{b_k},$$

where $a_k (\equiv k \pmod{\ell})$ and $b_k (\equiv k \pmod{r})$ for $k = 0, 1, \dots, \ell r - 1$. Then we can proceed analogously to **Case 4a** to obtain

$$\sum_{k=0}^{\ell r-1} m_k = \sum_{k=0}^{\ell r-1} (-i)^{a_k} i^{b_k} = \left(\sum_{k=0}^{\ell-1} (-i)^{a_k} \right) \left(\sum_{k=0}^{r-1} i^{b_k} \right) = i \times (-i) = 1$$

as required because $\ell \equiv r \equiv 3 \pmod{4}$.

Now clearly the car traverses through all points between $(0,0)$ and $(1,0)$ during the first section and, in fact, covers all points between $(n-1,0)$ and $(n,0)$ during the n -th section. Hence it will eventually reach $(d,0)$ for any positive d .

To summarize: (ℓ, r) satisfies the required conditions if and only if

$$\ell \equiv r \equiv 1 \quad \text{or} \quad \ell \equiv r \equiv 3 \pmod{4}.$$

□

Remark. In case $\gcd(\ell, r) = d \neq 1$, the answer is:

$$\frac{\ell}{d} \equiv \frac{r}{d} \equiv 1 \quad \text{or} \quad \frac{\ell}{d} \equiv \frac{r}{d} \equiv 3 \pmod{4}.$$